An Introduction to Quantum Spin Systems[1]
Notes for MA5020 (John von Neumann Guest Lectures)

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Contents

8. Symmetry, Excitation Spectrum, and Correlations

As before, let (Γ, d) be a discrete metric space with an F-function F. Let $\{\tau_t \mid t \in \mathbb{R}\}$ be the dynamics generated by an interaction $\Phi \in \mathcal{B}_F(\Gamma)$ for a quantum spin system on Γ. An automorphism α on $\mathcal{A}_\Gamma$ is called a *symmetry* of the C^*-dynamical system $(\mathcal{A}_\Gamma, \tau_t)$ if it commutes with the dynamics, i.e., $\tau_t \circ \alpha = \alpha \circ \tau_t$, for all $t \in \mathbb{R}$. It is easy to see that the symmetries of $(\mathcal{A}_\Gamma, \tau_t)$ form a subgroup of the automorphism group of $\mathcal{A}_\Gamma$.

Two important classes of symmetries are the so-called *point symmetries*, or lattice symmetries, and *gauge symmetries*. As an example of the first class, consider quantum spin systems on $\Gamma = \mathbb{Z}^\nu$, with the lattice distance d, and with $\mathcal{H}_x = \mathbb{C}^n$, for all $x \in \mathbb{Z}^\nu$. In general, $\mathcal{A}_\Gamma^{\rm loc}$ is generated as an algebra by the collection of single-site algebras $\mathcal{A}_{\{x\}}$, $x \in \Gamma$. Automorphisms of $\mathcal{A}_\Gamma$ are therefore uniquely determined by their action on the one-site algebras. In the situation at hand, $\mathcal{A}_{\{x\}} \cong M_n$, for all $x \in \mathbb{Z}^\nu$. Then, given these identifications, there is a unique automorphism θ_x on $\mathcal{A}_{\mathbb{Z}^\nu}$ that maps $\mathcal{A}_{\{0\}}$ into $\mathcal{A}_{\{x\}}$. The automorphisms θ_x is called the translation by x, and $x \mapsto \theta_x$ is a representation of the additive group $\mathbb{Z}^\nu$ into the automorphisms on $\mathcal{A}_{\mathbb{Z}^\nu}$. In the same way, any bijection of Γ can in principle be a symmetry of a quantum spin system defined on Γ. E.g., $\mathbb{Z}^\nu$, in addition to the lattice translations, also has rotation and reflection symmetries that can be represented on $\mathcal{A}_\Gamma$ as automorphisms.

Continuing with the important example of $\Gamma = \mathbb{Z}^\nu$, we will say that an interaction $\Phi : \mathcal{P}_0(\mathbb{Z}^\nu) \to \mathcal{A}_\Gamma^{\rm loc}$, is *translation invariant* if for all $x \in \mathbb{Z}^\nu$, and $X \in \mathcal{P}_0(\mathbb{Z}^\nu)$, $\theta_x(\Phi(X)) = \Phi(X + x)$. It is easy to see that if τ_t is defined in terms of a translation invariant interaction, then the translations will be symmetries of the dynamics, i.e., $\tau_t \circ \theta_x = \theta_x \circ \tau_t$, for $t \in \mathbb{R}, x \in \mathbb{Z}^\nu$.

The second class, the gauge symmetries, are generated by unitaries acting on $\mathcal{H}_x$. For any quantum spin system with algebra of quasi-local observables $\mathcal{A}_\Gamma$, suppose we have a unitary $u_x \in \mathcal{A}_{\{x\}}$, for all $x \in \Gamma$. It is then straightforward to show that there is a unique automorphism α on $\mathcal{A}_\Gamma$ such that $\alpha(A) = u_x^* A u_x$, for all $A \in \mathcal{A}_{\{x\}}$. Given a group G, if for all x we have a unitary representation $g \mapsto u_x(g)$ of a group G on $\mathcal{H}_x$, the corresponding α_g give a representation of G as automorphisms of $\mathcal{A}_\Gamma$.

The Heisenberg model introduced in (2.55) has a gauge symmetry given by a representation of $SU(2)$. For the spin 1/2 model, $n = 2$, and $SU(2)$ acts on $\mathbb{C}^2$ by its fundamental representation: for every $u \in SU(2)$, $A \in M_2$, $\alpha_u(A) = u^* A u$. For the spin-S Heisenberg model, the unitaries are given by the $2S+1$-dimensional irreducible representation of $SU(2)$. For a summary of the unitary representations of the Lie group $SU(2)$, see Appendix 2. Symmetries that are given by a representation of a Lie group (of dimension ≥ 1), are often referred to as continuous symmetries.

If α is a symmetry of $(\mathcal{A}_\Gamma, \tau_t)$, with the property that $\alpha(\mathcal{A}_\Gamma^{\rm loc}) \subset \mathcal{A}_\Gamma^{\rm loc}$, the generator δ of the dynamics $\tau_t = e^{it\delta}$ will also commute with α. Since under the above assumption, we have $\alpha(A) \in \mathrm{dom}(\delta)$, for all $A \in \mathcal{A}_\Gamma^{\rm loc}$, we can differentiate the identity $\alpha(\tau_t(A)) = \tau_t(\alpha(A))$ and obtain

$$\alpha \circ \tau_t(i\delta(A)) = \tau_t(i\delta(\alpha(A))). \tag{8.1}$$

After applying τ_{-t} to both sides of this equation and using the fact that α is a symmetry of the dynamics, we get

$$\alpha(\delta(A)) = \delta(\alpha(A)), \text{ for all } A \in \mathcal{A}_\Gamma^{\rm loc}. \tag{8.2}$$

Note that both point symmetries and gauge symmetries leave $\mathcal{A}_\Gamma^{\rm loc}$ invariant, hence (1.2) holds for these types of symmetries. Using this property it is easy to verify that the sets $\mathcal{S}_\beta$, $\beta \in [0, \infty]$, define in Section 7.2 will be invariant under such symmetries of $(\mathcal{A}_\Gamma, \tau_t)$. That is, $\mathcal{S}_\beta$ is invariant as a set, meaning

$$\mathcal{S}_\beta = \{\omega \circ \alpha \mid \omega \in \mathcal{S}_\beta\}. \tag{8.3}$$

If, moreover, we have $\omega \circ \alpha = \omega$, for all $\omega \in \mathcal{S}_\beta$, we say that the symmetry α is unbroken. In the opposite case, i.e., if there exists $\omega \in \mathcal{S}_\beta$ such that $\omega \circ \alpha \neq \omega$, we say that the symmetry α is *spontaneously broken.* Spontaneous symmetry breaking at some $\beta < \infty$ signals a phase transition. Spontaneous symmetry at any value of β has important physical consequences and it is a central concept to many phenomena in condensed matter and particle physics.

In the next section we will consider systems on $\mathbb{Z}^\nu$ with $\mathcal{H}_x = \mathbb{C}^n$, for all $x \in \mathbb{Z}^\nu$ and a continuous local symmetry that commutes with translations. Explicitly, this means that for all $s \in \mathbb{R}$, we α_s is of the form

$$\alpha_s(A) = \left(\bigotimes_{x\in X} u(s)^*\right) A \left(\bigotimes_{x\in X} u(s)\right), \text{ for all } A \in \mathcal{A}_X, \tag{8.4}$$

where $u(s) = e^{isJ}$, with $J = J^* \in M_n$. There may be more than one such continuous family of α_s. The $SU(2)$ symmetry of the Heisenberg model is an example of this situation.

Both the notion of gapped excitation spectrum and the concept of spontaneous symmetry breaking are most conveniently formulated for infinite systems. The spectrum of a finite quantum spin system is a finite set of eigenvalues. One usually associates the smallest eigenvalue with the ground states, but in some cases it is appropriate to consider one or more additional eigenvalues as corresponding to 'the ground state' as well. Also, there is no simple relationship between the degeneracy of the ground state eigenvalue and the number of distinct ground states in the thermodynamic limit. A illustrative example is the AKLT chain, which has a four-dimensional eigenspace belonging to the ground state eigenvalue, yet, there is a unique thermodynamic limit of those states [1]. Numerical calculations have shown that the spin-1 antiferromagnetic Heisenberg chain has a pair of eigenvalues corresponding to a very similar four-dimensional space of states that converge to a unique ground state in the thermodynamic limit [15]. For similar reasons, spontaneous symmetry breaking, which leads to multiple ground states in the thermodynamic limit, may be 'hidden' in a unique ground state in finite volume. This occurs in the antiferromagnetic Heisenberg models on finite volumes of even size in two and more dimensions [16].

All these ambiguities disappear in the thermodynamic limit. E.g., the excitation spectrum above a ground state ω is simply the spectrum of its GNS Hamiltonian H_ω, which is non-negative and the ground state is represented by an eigenvector with eigenvalue 0. The spectral gap γ can then be defined as follows:

$$\gamma = \sup\{\delta > 0 \mid (0,\delta) \cap \mathrm{spec} H_\omega = \emptyset\}, \tag{8.5}$$

with the convention that we put $\gamma = 0$ if the RHS is the empty set.

8.1. **The Goldstone Theorem.** The Goldstone's theorem in quantum field theory shows that the spontaneous breaking of a continuous symmetry breaking is always accompanied by the appearance of a massless particle, i.e., a gapless excitation or Goldstone mode [8,9,14]. As we explained before, spontaneous symmetry breaking is indicated by the existence of $\omega \in \mathcal{S}_\beta$ that are *not* invariant under the symmetry. In statistical mechanics, spontaneous breaking of a continuous symmetry in the ground state of a translation invariant system also implies a gapless excitation spectrum, while at positive temperature the symmetry breaking precludes fast (integrable) decay of correlations [17].

The Goldstone Theorem requires translation invariance. In order to keep the presentation of the basic argument here as simple as possible, we will limit the discussion to translation invariant quantum spin systems $\mathcal{A}_\Gamma$ with $\Gamma = \mathbb{Z}^\nu$ for some $\nu \geq 1$. It is possible to generalize Theorem 1.1: e.g. more general lattices can be considered, and it is also not crucial to assume full translation invariance. See the remarks following the proof of Theorem 1.1 for pointers to several generalizations.

Let $(\mathcal{A}_{\mathbb{Z}^\nu}, \tau_t)$ be a C^*-dynamical system with dynamics τ_t generated by $\Phi \in \mathcal{B}_F(\mathbb{Z}^\nu)$. Note: To be precise, translation invariance will be assumed of the ground state, however, the interaction Φ need not be translation invariant. Let α_s be a continuous, locally-generated symmetry on $\mathcal{A}_{\mathbb{Z}^\nu}$. In particular, this means that there is $J_0^* = J_0 \in \mathcal{A}_{\{0\}}$ and for any $x \in \mathbb{Z}^\nu$, one defines $J_x = \theta_x(J_0) \in$

$\mathcal{A}_{\{x\}}$ where θ_x denotes translation by x. In terms of these local operators, α_s is the one-parameter group of automorphisms of $\mathcal{A}_{\mathbb{Z}^\nu}$ defined by

$$\alpha_s(A) = e^{is\sum_x J_x} A e^{-is\sum_x J_x} \quad \text{for all } A \in \mathcal{A}_{\mathbb{Z}^\nu} \text{ and } s \in \mathbb{R}. \tag{8.6}$$

Let ω be a translation invariant ground state of $(\mathcal{A}_{\mathbb{Z}^\nu}, \tau_t)$, i.e. ω satisfies the ground state condition

$$\omega(A^*\delta(A)) \geq 0 \quad \text{for all } A \in D(\delta) \tag{8.7}$$

and also translation invariance

$$\omega(\theta_x(A)) = \omega(A) \quad \text{for every } A \in \mathcal{A}_{\mathbb{Z}^\nu} \text{ and } x \in \mathbb{Z}^\nu \tag{8.8}$$

In one formulation of the Goldstone's theorem one assumes that a translation-invariant ground state satisfies the following inequalities for a constant $\gamma > 0$:

$$\gamma\omega(A^*A) \leq \omega(A^*\delta(A)) \quad \text{for all } A \in \mathcal{D}(\delta) \text{ with } \omega(A) = 0\,, \tag{8.9}$$

This property implies that (i) ω is a ground state; (ii) that $\ker H_\omega = \mathbb{C}\mathbb{1}$; and (iii) there is a gap in the spectrum of H_ω above the ground state. If the ground state is unique, this property is actually *equivalent* to the existence of a spectral of size $\geq \gamma$. The theorem then states that these assumptions imply that the continuous symmetry is unbroken, which is the contrapositive of the more frequently encountered statement that continuous symmetry breaking implies gapless excitations (massless particles).

Theorem 8.1 (Goldstone for ground-states [17])**.** *Let $(\mathcal{A}_{\mathbb{Z}^\nu}, \tau_t)$ be C^*-dynamical system over $\mathbb{Z}^\nu$ with τ_t generated by an interaction $\Phi \in \mathcal{B}_F(\mathbb{Z}^\nu)$. Let α_s be a continuous gauge symmetry of the dynamics, i.e., α_s is of the form (**??**)*

$$\alpha_s \circ \tau_t = \tau_t \circ \alpha_s \quad \textit{for all } s, t \in \mathbb{R}\,. \tag{8.10}$$

Let ω be a translation invariant ground state of $(\mathcal{A}_{\mathbb{Z}^\nu}, \tau_t)$, for which there exists $\gamma > 0$ such that

$$\gamma \cdot \omega(A^*A) \leq \omega(A^*\delta(A)) \quad \textit{for all } A \in \mathcal{A}_\Gamma^{\text{loc}} \textit{ with } \omega(A) = 0\,. \tag{8.11}$$

Then, ω is invariant with respect to α_s, i.e.

$$\omega(\alpha_s(A)) = \omega(A) \quad \textit{for all } A \in \mathcal{A}_{\mathbb{Z}^\nu} \textit{ and all } s \in \mathbb{R}. \tag{8.12}$$

Before we start the proof of this theorem, we derive a lemma that exploits the assumption that the interaction Φ has a finite F-norm.

Lemma 8.2. *Consider a C^*-dynamical system $(\mathcal{A}_{\mathbb{Z}^\nu}, \tau_t)$ with dynamics $\tau_t = e^{it\delta}$ generated by $\Phi \in B_F(\mathbb{Z}^\nu)$. For any $A \in \mathcal{A}_{\mathbb{Z}^\nu}^{\text{loc}}$,*

$$\sup_{x \in \mathbb{Z}^\nu} \sum_{y \in \mathbb{Z}^\nu} \|[\theta_x(A), \delta(\theta_y(A))]\| \leq 4\|A\|^2 \|F\| \|\Phi\|_F |X| (2\text{diam}(X) + 1)^\nu, \tag{8.13}$$

and moreover, for $d > \text{diam}(X)$, we have the esitmate

$$\sup_{x \in \mathbb{Z}^\nu} \sum_{\substack{y \in \mathbb{Z}^\nu: \\ |y-x| \geq d}} \|[\theta_x(A), \delta(\theta_y(A))]\| \leq 4\|A\|^2 \|\Phi\| |X|^2 \sum_{\substack{y' \in \mathbb{Z}^\nu: \\ |y'| \geq d - \text{diam}(X)}} F(|y'|), \tag{8.14}$$

which implies that the quantity vanishes as $d \to \infty$.

Proof. Let $A \in \mathcal{A}_{\mathbb{Z}^\nu}^{\text{loc}}$ and denote by $X = \text{supp}(A) \in \mathcal{P}_0(\mathbb{Z}^\nu)$. To ease notation, for any $y \in \mathbb{Z}^\nu$ we will set

$$A_y = \theta_y(A) \in \mathcal{A}_{X_y} \quad \text{where} \quad X_y = \{x + y : x \in X\} \in \mathcal{P}_0(\mathbb{Z}^\nu)\,. \tag{8.15}$$

Observe that for any fixed $x \in \mathbb{Z}^\nu$

$$\sum_{y\in\mathbb{Z}^\nu} \|[A_x, \delta(A_y)]\| \le \sum_{y\in\mathbb{Z}^\nu} \sum_{\substack{Z\in\mathcal{P}_0(\mathbb{Z}^\nu):\\ Z\cap X_y\neq\emptyset}} \|[A_x,[\Phi(Z),A_y]]\| \tag{8.16}$$

For those $y \in \mathbb{Z}^\nu$ such that $X_y \cap X_x \neq \emptyset$, we estimate the above as

$$\begin{aligned}
\sum_{\substack{y\in\mathbb{Z}^\nu:\\ X_y\cap X_x\neq\emptyset}} \sum_{\substack{Z\in\mathcal{P}_0(\mathbb{Z}^\nu):\\ Z\cap X_y\neq\emptyset}} \|[A_x,[\Phi(Z),A_y]]\| &\le 4\|A\|^2 \sum_{\substack{y\in\mathbb{Z}^\nu:\\ X_y\cap X_x\neq\emptyset}} \sum_{z\in X_y} \sum_{\substack{Z\in\mathcal{P}_0(\mathbb{Z}^\nu):\\ z\in Z}} \|\Phi(Z)\| \\
&\le 4\|A\|^2\|F\|\|\Phi\|_F|X|(2\mathrm{diam}(X)+1)^\nu, \qquad (8.17)
\end{aligned}$$

where we estimated the number of y such that $X_y \cap X_x \neq \emptyset$ by the cardinality of the ball, centered at the origin, having radius equal to the diameter of X.

For the remaining y, we have

$$\begin{aligned}
\sum_{\substack{y\in\mathbb{Z}^\nu:\\ X_y\cap X_x=\emptyset}} \sum_{\substack{Z\in\mathcal{P}_0(\mathbb{Z}^\nu):\\ Z\cap X_y\neq\emptyset}} \|[A_x,[\Phi(Z),A_y]]\| &\le 4\|A\|^2 \sum_{\substack{y\in\mathbb{Z}^\nu:\\ X_y\cap X_x=\emptyset}} \sum_{z_1\in X_y} \sum_{z_2\in X_x} \sum_{\substack{Z\in\mathcal{P}_0(\mathbb{Z}^\nu):\\ z_1,z_2\in Z}} \|\Phi(Z)\| \\
&\le 4\|A\|^2\|\Phi\|_F \sum_{\substack{y\in\mathbb{Z}^\nu:\\ X_y\cap X_x=\emptyset}} \sum_{z_1\in X_y} \sum_{z_2\in X_x} F(|z_1-z_2|) \\
&\le 4\|A\|^2\|\Phi\|_F \sum_{x_1\in X} \sum_{x_2\in X} \sum_{y\in\mathbb{Z}^\nu} F(|x_1-x_2+y-x|) \\
&\le 4\|A\|^2\|\Phi\|_F\|F\||X|^2 \qquad (8.18)
\end{aligned}$$

Both estimates in (1.17) and (1.18) are independent of x, and it is straightforward to show they both are bounded above as in (1.13).

The proof of (1.14) is similar. In fact, let A be as above and take $d > \mathrm{diam}(X)$. In this case, any y with $|x-y| \ge d$ satisfies $X_x \cap X_y = \emptyset$. Hence, estimating as in (1.18) we find that

$$\begin{aligned}
\sum_{\substack{y\in\mathbb{Z}^\nu:\\ |y-x|\ge d}} \|[A_x,\delta(A_y)]\| &\le 4\|A\|^2\|\Phi\|_F \sum_{\substack{y\in\mathbb{Z}^\nu:\\ |y-x|\ge d}} \sum_{x_1\in X} \sum_{x_2\in X} F(|x_1-x_2+y-x|) \\
&\le 4\|A\|^2\|\Phi\||X|^2 \sum_{\substack{y'\in\mathbb{Z}^\nu:\\ |y'|\ge d-\mathrm{diam}(X)}} F(|y'|) \qquad (8.19)
\end{aligned}$$

Since F is uniformly integrable, this proves (1.14). □

Proof of Theorem 1.1. We will prove that $\omega\circ\alpha_s = \omega$, for all $s \in \mathbb{R}$ by showing that, for all $A \in \mathcal{A}_\Gamma$

$$\frac{d}{ds}\omega(\alpha_s(A)) = 0, \text{ for all } s \in \mathbb{R}. \tag{8.20}$$

Since the strictly local observables are dense in $\mathcal{A}_\Gamma$, it suffices to show this for $A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$. For local A, say $A \in \mathcal{A}_X$, $X \in \mathcal{P}_0(\Gamma)$, we have

$$\frac{d}{ds}\omega(\alpha_s(A)) = i\omega([J_x,(\alpha_s(A))]). \tag{8.21}$$

Since α_s is an automorphism leaving $\mathcal{A}_X$ invariant, it follows that it is sufficient to show

$$\omega([J_X,A)]) = 0, \text{ for all } X \in \mathcal{P}_0(\Gamma), \text{ and } A \in \mathcal{A}_X. \tag{8.22}$$

We prove (1.22) in two steps. First, by averaging over translations, we establish an upper bound, see (1.25) below. Next, we analyze the factors in our upper bound and see that this bound goes to zero as we average over more and more translations.

As a consequence of the translation invariance of ω and the commutation of the translations and the symmetry, for any finite $\Lambda \subset \mathbb{Z}^\nu$, we have that

$$\omega([J_X, A)]) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega([J_{X_x}, \theta_x(A)]) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega([J_\Lambda, \theta_x(A)]) = \frac{1}{|\Lambda|}\omega([J_\Lambda, A_\Lambda]), \tag{8.23}$$

where we have set

$$A_\Lambda = \sum_{x\in\Lambda}\theta_x(A), \text{ and } J_\Lambda = \sum_{x\in \mathrm{supp}(A_\Lambda)} J_x\,. \tag{8.24}$$

We will now prove the upper bound

$$|\omega([J_X, A)])|^2 = \frac{1}{|\Lambda|^2}|\omega([J_\Lambda, A_\Lambda])|^2 \leq \frac{4}{\gamma^2}\cdot\frac{1}{|\Lambda|}\omega([J_\Lambda, \delta(J_\Lambda)])\cdot\frac{1}{|\Lambda|}\omega([A_\Lambda, \delta(A_\Lambda)]) \tag{8.25}$$

valid for any $\Lambda \in \mathcal{P}_0(\mathbb{Z}^\nu)$, and $\gamma > 0$ is the constant appearing the in the assumptions of the theorem.

To prove (1.25), we start by noting that due to the commutator

$$\omega([J_\Lambda, A_\Lambda]) = \omega([\hat{J}_\Lambda, \hat{A}_\Lambda]) \tag{8.26}$$

where

$$\hat{B} = B - \omega(B)\mathbb{1} \quad \text{for any } B \in \mathcal{A}_{\mathbb{Z}^\nu} \tag{8.27}$$

Next, using Cauchy-Schwartz of both observables, we get

$$|\omega([J_\Lambda, A_\Lambda])|^2 \leq 4\omega(\hat{J}_\Lambda^*\hat{J}_\Lambda)\cdot\omega(\hat{A}_\Lambda^*\hat{A}_\Lambda) \tag{8.28}$$

For any local observable B, the time-invariance of the state implies

$$\omega(\delta(B)B^*) + \omega(B\delta(B^*)) = \omega(\delta(BB^*)) = 0. \tag{8.29}$$

For any local observable B, $\omega(\hat{B}) = 0$. Using the assumption (1.11), (1.29), and the ground state property, we obtain

$$\begin{aligned} \omega(\hat{B}^*\hat{B}) \leq \frac{1}{\gamma}\omega(\hat{B}^*\delta(\hat{B})) &= \frac{1}{\gamma}\omega([\hat{B}^*, \delta(\hat{B})]) + \omega(\delta(\hat{B})\hat{B}^*) \\ &= \frac{1}{\gamma}\omega([\hat{B}^*, \delta(\hat{B})]) - \omega(\hat{B}\delta(\hat{B}^*)) \\ &\leq \frac{1}{\gamma}\omega([\hat{B}^*, \delta(\hat{B})]) = \frac{1}{\gamma}\omega([B^*, \delta(B)]) \end{aligned} \tag{8.30}$$

Using this bound for $B = J_\Lambda$ and $B = A_\Lambda$ in (1.28) yields (1.25) as claimed.

For the second step in the proof, we use Lemma 1.2 to further estimate the right-hand-side of (1.25). For the second factor in the right-hand-side of (1.25), note that

$$\begin{aligned} \frac{1}{|\Lambda|}\omega([A_\Lambda, \delta(A_\Lambda)]) &\leq \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{y\in\Lambda}|\omega\left([\theta_x(A), \delta(\theta_y(A))]\right)| \\ &\leq \sup_{x\in\mathbb{Z}^\nu}\sum_{y\in\mathbb{Z}^\nu}\|[\theta_x(A), \delta(\theta_y(A))]\| \end{aligned} \tag{8.31}$$

which is finite by Lemma 1.2. This bound is independent of the set Λ and the support of the observable A.

Finally, we argue that the first factor in the right-hand-side of (1.25) tends to zero for a suitable choice of sets Λ which grow to $\mathbb{Z}^\nu$. To prove this we will need to use the assumption that the dynamics and the symmetry commute, i.e. (1.10).

Let us denote by $\mathcal{J}_\Lambda = \mathrm{supp}(J_\Lambda)$. It is clear that

$$\omega([J_\Lambda, \delta(J_\Lambda)]) = \sum_{x,y\in\mathcal{J}_\Lambda} \omega([J_x, \delta(J_y)]) = \sum_{x,y\in\mathbb{Z}^\nu} \chi_{\mathcal{J}_\Lambda}(x)\chi_{\mathcal{J}_\Lambda}(y)\omega([J_x, \delta(J_y)]) \tag{8.32}$$

with $\chi_{\mathcal{J}_\Lambda}$ being a characteristic function. Note that by using again the time-invariance of ω, we have $\omega(J_x\delta(J_y)) = -\omega(\delta(J_x)J_y)$. Therefore,

$$\omega([J_x, \delta(J_y)]) = \omega([J_y, \delta(J_x)]) \tag{8.33}$$

We now apply (1.2) to the observables J_y and once more the time-invariance of ω:

$$\omega(\alpha_s(\delta(J_y)) = \omega(\delta(\alpha_s(J_y)) = 0. \tag{8.34}$$

By taking the derivative with respect to s, in $s = 0$, of this equation, we obtain

$$\sum_{x\in\mathbb{Z}^\nu} \omega(J_x, \delta(J_y)]) = 0, \text{ for all } y \in \mathbb{Z}^\nu. \tag{8.35}$$

Note that Lemma 1.2 guarantees that the sum above is absolutely summable. Now, using (1.33) and (1.35), we can re-write (1.32) as

$$\omega([J_\Lambda, \delta(J_\Lambda)]) = -\frac{1}{2} \sum_{x,y\in\mathbb{Z}^\nu} |\chi_{\mathcal{J}_\Lambda}(x) - \chi_{\mathcal{J}_\Lambda}(y)|^2 \omega([J_x, \delta(J_y)]) \tag{8.36}$$

from which we derive the following bound

$$|\omega([J_\Lambda, \delta(J_\Lambda)])| \leq \sum_{x\in\mathcal{J}_\Lambda} \sum_{y\in\mathcal{J}_\Lambda^c} |\omega([J_x, \delta(J_y)])|. \tag{8.37}$$

Up to this point, the support of the observables being considered has played a minor role. To complete the argument, we now make this more explicit. Recall that the original self-adjoint observable A was assumed local. Let us denote by $X = \mathrm{supp}(A)$. Without loss of generalty, we will assume that X is a cube with side-length $\ell > 0$, hence $|X| = (\ell + 1)^\nu$. It is convenient to choose the finite set Λ, as in (1.25) above, to also be a cube, e.g. with side-length $L > 0$. In this case,

$$\mathcal{J}_\Lambda = \mathrm{supp}(J_\Lambda) = \mathrm{supp}(A_\Lambda) \quad \text{satisfies} \quad |\mathcal{J}_\Lambda| = (L + \ell + 1)^\nu . \tag{8.38}$$

Consider $L \geq 1$ large and fix an integer d, independent of L, satisfying $\ell < 2d < L$. Set

$$\mathcal{J}_\Lambda^{\mathrm{int}} = \{x \in \mathcal{J}_\Lambda : \mathrm{dist}(x, \partial\mathcal{J}_\Lambda) \geq d\} \quad \text{and} \quad \mathcal{J}_\Lambda^{\mathrm{bd}} = \mathcal{J}_\Lambda \setminus \mathcal{J}_\Lambda^{\mathrm{int}} . \tag{8.39}$$

It is clear that

$$\sum_{x\in\mathcal{J}_\Lambda^{\mathrm{int}}} \sum_{y\in\mathcal{J}_\Lambda^c} |\omega([J_x, \delta(J_y)])| \leq |\mathcal{J}_\Lambda^{\mathrm{int}}| \cdot f(d) \quad \text{where} \quad f(d) = \sup_{x\in\mathbb{Z}^\nu} \sum_{\substack{y\in\mathbb{Z}^\nu: \\ |x-y|\geq d}} \|[J_x, \delta(J_y)]\| \tag{8.40}$$

For the remaining boundary terms, we have that

$$\sum_{x\in\mathcal{J}_\Lambda^{\mathrm{bd}}} \sum_{y\in\mathcal{J}_\Lambda^c} |\omega([J_x, \delta(J_y)])| \leq |\mathcal{J}_\Lambda^{\mathrm{bd}}| \cdot C \quad \text{with} \quad C = \sup_{x\in\mathbb{Z}^\nu} \sum_{y\in\mathbb{Z}^\nu} \|[J_x, \delta(J_y)]\| \tag{8.41}$$

C is guaranteed to be finite due Lemma 1.2. Combining these estimates, we have then that

$$\limsup_{L\to\infty} \frac{1}{|\Lambda|} |\omega([J_\Lambda, \delta(J_\Lambda)])| \leq f(d) \tag{8.42}$$

Taking now $d \to \infty$, we complete the proof. $\square$

The set of ground states of the spin-1/2 ferromagnetic Heisenberg model on $\mathbb{Z}^\nu$ is spanned by the translation invariant pure product states, i.e., states determined by a unit vector $\phi \in \mathbb{C}^2$ through the formula,

$$\omega(A) = \langle \bigotimes_{x\in X} \phi, A \bigotimes_{x\in X} \phi \rangle, \text{ for all } , X \in \mathcal{P}_0(\mathbb{Z}^\nu), A \in \mathcal{A}_X. \tag{8.43}$$

Clearly, the $SU(2)$ symmetry of the model is spontaneously broken in the ground state. To illustrate the Goldstone Theorem, take for example $\phi = |+\rangle$, the unit eigenvector of σ^3 with eigenvalue 1. The GNS representation is easy to construct with $\mathcal{H}_\omega$ the separable Hilbert space with orthonormal basis $\{\xi_X \mid X \in \mathcal{P}_0(\mathbb{Z}^\nu)\}$. $\Omega_\omega = \xi_\emptyset$ and can be thought of as the formal tensor product $\bigotimes_{x\in\mathbb{Z}^\nu} |+\rangle$ and the representation π_ω can be constructed starting from

$$\pi_\omega(\sigma^-)\xi_X = \begin{cases} \xi_{X\cup\{x\}} & \text{if } x \notin X \\ 0 & \text{if } x \in X \end{cases}$$

It is then straightforward to show that $\mathcal{H}_1 = \text{span}\{\xi_{\{x\}} \mid x \in \mathbb{Z}^n u\}$ is an invariant subspace for H_ω. A simple calculation shows that $\text{spec}(H_\omega \restriction_{\mathcal{H}_1}) = [0, c]$, with $c > 0$, and hence H_ω is gapless.

8.2. **The Exponential Clustering Theorem.** let $(\mathcal{A}_\Gamma, \tau_t)$ be a C^*-dynamical system with $\mathcal{A}_\Gamma$ the quasi-local algebra corresponding to a quantum spin system over Γ and τ_t the dynamics generated by an interaction $\Phi \in \mathcal{B}_{F_a}(\Gamma)$, where $F_a(r) = e^{-ar}F(r)$, and $a > 0$.

The Exponential Clustering Theorem concerns a situation complementary to the gapless states associated with spontaneous breaking of a continuous symmetry considered, as implied by the Goldstone Theorem of the previous section. Briefly, The Exponential Clustering Theorem states that if there is a spectral gap above a ground states, then correlations in this state decay exponentially fast with the distance. The natural setting for this type of result is again systems on a infinite set (Γ, d). Translation invariance plays no direct role, however, and is not a condition of the theorem.

More precisely, let $(\mathcal{A}_\Gamma, \tau_t)$ be a C^*-dynamical system with $\mathcal{A}_\Gamma$ the quasi-local algebra corresponding to a quantum spin system over Γ and τ_t the dynamics generated by an interaction $\Phi \in \mathcal{B}_{F_a}(\Gamma)$. Assume that this system can be represented by a Hamiltonian H on a Hilbert space $\mathcal{H}$. This means that we assume there is a representation $\pi : \mathcal{A}_\Gamma \to \mathcal{B}(\mathcal{H})$ and a (densely defined) self-adjoint operator H on $\mathcal{H}$ for which

$$\pi(\tau_t(A)) = e^{itH}\pi(A)e^{-itH} \quad \text{for all} \quad A \in \mathcal{A}_\Gamma \text{ and } t \in \mathbb{R}. \tag{8.44}$$

We will further assume that $H \geq 0$ and has a spectral gap $\gamma > 0$ above 0, as defined in (1.5). Let P_0 denote the orthogonal projection onto $\ker H$. For the remainder of this section, we will always work in this representation and we simplify the notation by writing A instead of $\pi(A)$.

Theorem 8.3 (Exponential Clustering)**.** *Let $a > 0$ and take $\Phi \in \mathcal{B}_{F_a}(\Gamma)$. Suppose that the dynamics corresponding to Φ on Γ can be represented by a Hamiltonian H with a gap $\gamma > 0$ above the ground state energy, as described above. Let Ω be a normalized ground state vector for H; i.e. satisfy $H\Omega = 0$ with $\|\Omega\| = 1$. Then, there exists a constant $\mu > 0$ such that for any local observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $X, Y \subset V$ and $d(X, Y) > 0$ satisfying $P_0B\Omega = P_0B^*\Omega = 0$, the bound*

$$|\langle \Omega, A\tau_{ib}(B)\Omega\rangle| \;\leq\; C(A, B, \gamma)\, e^{-\mu d(X,Y)\left(1+\frac{\gamma^2 b^2}{4\mu^2 d(X,Y)^2}\right)} \tag{8.45}$$

is valid for all non-negative b satisfying $0 \leq b\gamma \leq 2\mu d(X, Y)$. One may take

$$\mu \;=\; \frac{a\,\gamma}{4\|\Phi\|_a C_a \,+\, \gamma}, \tag{8.46}$$

as well as a constant

$$C(A,B,\gamma) \,=\, \|A\|\,\|B\| \left[1 \,+\, \sqrt{\frac{1}{\mu d(X,Y)}} \,+\, \frac{2\|F_0\|}{\pi C_a} \min\left(|\partial_\Phi X|, |\partial_\Phi Y|\right) \right]. \tag{8.47}$$

Note that in the case of a non-degenerate ground state, the condition on B is equivalent to $\langle \Omega, B\Omega \rangle = 0$. In this case, the theorem with $b = 0$ becomes

$$|\langle \Omega, AB\Omega \rangle - \langle \Omega, A\Omega \rangle \langle \Omega, B\Omega \rangle| \leq C(A,B,\gamma)\, e^{-\mu d(X,Y)}, \tag{8.48}$$

which is the standard (equal-time) correlation function. For small $b > 0$, the estimate (1.45) can be viewed as a perturbation of (1.48). Moreover, for $b > 0$ large, there is a trivial bound

$$|\langle \Omega, A\tau_{ib}(B)\Omega \rangle| \,\leq\, \|A\|\,\|B\|\, e^{-b\gamma}. \tag{8.49}$$

Proof. The proof of this result has two main steps. First, using techniques from complex analysis, we reduce an estimate on the quantity of interest to that of an integral over the real line. Next, we carefully analyze the resulting integral.

Step 1: We begin by noting that for any $z \in \mathbb{C}$ with $\mathrm{Im}[z] \geq 0$, the function f with

$$f(z) \,:=\, \langle \Omega, A\tau_z(B)\Omega \rangle \,=\, \int_\gamma^\infty e^{izE}\, d\,\langle A^*\Omega, P_E B\Omega \rangle \tag{8.50}$$

The final equality above uses the spectral theorem for the self-adjoint operator H and the fact that that B projects off the ground state, i.e. $P_0 B\Omega = 0$. This integral representation of f clearly demonstrates that f is analytic in $\mathbb{C}^+$, i.e. those $z \in \mathbb{C}$ with $\mathrm{Im}[z] > 0$, and moreover, f has a continuous (and bounded) boundary value on the real axis. Our first goal is to now estimate $|f(ib)|$ for $b > 0$ as in the statement above. The case $b = 0$ will follow by a limiting argument.

Let $b > 0$. For any $T > b$, denote by Γ_T the semi-circular contour in the upper-half plane which passes through the points $-T$, T, and iT. A simple limiting argument, using that f has a continuous boundary value, shows that

$$f(ib) \,=\, \frac{1}{2\pi i} \int_{\Gamma_T} \frac{f(z)}{z - ib} dz\,. \tag{8.51}$$

The first step in the proof is completed when we demonstrate that the piece of this contour integral that extends into the upper half plane is vanishingly small, i.e. we claim that

$$|\langle \Omega, A\tau_{ib}(B)\Omega \rangle| \,=\, |f(ib)| \,\leq\, \limsup_{T\to\infty} \left| \frac{1}{2\pi i} \int_{-T}^{T} \frac{f(t)}{t - ib} dt \right|. \tag{8.52}$$

This follows from two observations. First, given (1.50), it is clear that

$$|f(e^{i\theta}T)| \leq \|A\|\|B\| e^{-T\gamma\sin(\theta)} \quad \text{for all } \theta \in [0,\pi]\,. \tag{8.53}$$

Lastly, the integral over the arc, e.g. when $T > 2b$, can be bounded by

$$\frac{\|A\|\|B\|}{\pi} \int_0^\pi e^{-T\gamma\sin(\theta)}\, d\theta \tag{8.54}$$

which clearly goes to zero as $T \to \infty$ by dominated convergence. This proves (1.52). We note that while (1.52) is true for any value of $b > 0$, we will have to choose $b > 0$ sufficiently small later in the proof; see the comments following (1.70) below.

Step 2: We now estimate the integral over the real line. Let $\alpha > 0$; this is a free parameter which will be judiciously chosen later. It is convenient to write

$$f(t) \,=\, e^{-\alpha b^2} \left[f(t)e^{-\alpha t^2} \,+\, f(t)\left(e^{\alpha b^2} - e^{-\alpha t^2}\right) \right]. \tag{8.55}$$

With this in mind, the pre-limit integral on the right-hand-side of (1.52) can be estimated by

$$e^{-\alpha b^2}\left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{f(t)e^{-\alpha t^2}}{t-ib}dt\right| + e^{-\alpha b^2}\left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{f(t)\left(e^{\alpha b^2}-e^{-\alpha t^2}\right)}{t-ib}dt\right|. \tag{8.56}$$

We will bound the absolute value of each of the integrals appearing in (1.56) separately; the prefactor $e^{-\alpha b^2}$ will be an additional damping made explicit by the choice of α.

To bound the first integral appearing in (1.56), we further divide the integrand into two terms. Note that

$$f(t)\,e^{-\alpha t^2} = \langle\Omega,\tau_t(B)A\Omega\rangle e^{-\alpha t^2} + \langle\Omega,[A,\tau_t(B)]\,\Omega\rangle e^{-\alpha t^2}. \tag{8.57}$$

Appealing again to the spectral theorem, we have that

$$\frac{1}{2\pi i}\int_{-T}^{T}\frac{\langle\Omega,\tau_t(B)A\Omega\rangle\,e^{-\alpha t^2}}{t-ib}dt = \int_{\gamma}^{\infty}\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{-itE}e^{-\alpha t^2}}{t-ib}dt\,d\langle P_E B^*\Omega, A\Omega\rangle, \tag{8.58}$$

using that $P_0B^*\Omega = 0$ as well. An application of Lemma 1.4, stated below, now yields that

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{-itE}\,e^{-\alpha t^2}}{t-ib}dt = \frac{1}{2\sqrt{\pi\alpha}}\int_0^{\infty}e^{-bw}e^{-\frac{(w+E)^2}{4\alpha}}dw \le \frac{1}{2}e^{-\frac{\gamma^2}{4\alpha}}, \tag{8.59}$$

since each of $E \ge \gamma > 0$, $\alpha > 0$, and $b > 0$ hold. Altogether, this proves that

$$\limsup_{T\to\infty}\left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{\langle\Omega,\tau_t(B)A\Omega\rangle\,e^{-\alpha t^2}}{t-ib}dt\right| \le \frac{\|A\|\,\|B\|}{2}e^{-\frac{\gamma^2}{4\alpha}}. \tag{8.60}$$

For the integral corresponding to the second term in (1.57), we use the bound

$$\left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{\langle\Omega,[A,\tau_t(B)]\,\Omega\rangle\,e^{-\alpha t^2}}{t-ib}dt\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\|[A,\tau_t(B)]\|}{|t|}e^{-\alpha t^2}dt, \tag{8.61}$$

which takes advantage of the fact that $b > 0$. To complete the estimate of the integral on the right-hand-side of (1.61), we introduce another free parameter $s > 0$. For values of $|t| \le s$, we estimate with the Lieb-Robinson bound, i.e. Theorem **??**, and for $|t| > s$ the gaussian factor dominates the integral. A short calculation shows that the right-hand-side of (1.61) is bounded from above by

$$\frac{2\,\|A\|\,\|B\|}{\pi\,\|\Phi\|_a\,C_a}D_a(X,Y)\left(e^{2\|\Phi\|_a C_a s}-1\right) + \frac{\|A\|\,\|B\|}{s\sqrt{\pi\alpha}}e^{-\alpha s^2}. \tag{8.62}$$

This completes the bound of the first integral appearing in (1.56) in terms of two free parameters $s > 0$ and $\alpha > 0$.

We now turn to the second integral in (1.56) and claim that if $2\alpha b \le \gamma$, then

$$\limsup_{T\to\infty}\left|\frac{1}{2\pi i}\int_{-T}^{T}\frac{f(t)\left(e^{\alpha b^2}-e^{-\alpha t^2}\right)}{t-ib}dt\right| \le \frac{\|A\|\,\|B\|}{2}e^{-\frac{\gamma^2}{4\alpha}}. \tag{8.63}$$

To prove this, first insert (1.50) into the integrand above and see that

$$\frac{1}{2\pi i}\int_{-T}^{T}\frac{f(t)\left(e^{\alpha b^2}-e^{-\alpha t^2}\right)}{t-ib}dt = \int_{\gamma}^{\infty}\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{itE}\left(e^{\alpha b^2}-e^{-\alpha t^2}\right)}{t-ib}dt\,d\langle A^*\Omega, P_E B\Omega\rangle \tag{8.64}$$

The inner integral on the right-hand-side above can be re-written

$$\begin{aligned}\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{itE}\left(e^{\alpha b^2}-e^{-\alpha t^2}\right)}{t-ib}dt &= e^{\alpha}b^2 H_T(E;ib)-\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{itE}\,e^{-\alpha t^2}}{t-ib}dt \\ &= e^{\alpha b^2}e^{-Eb}-\frac{1}{2\sqrt{\pi\alpha}}\int_0^{\infty}e^{-bw}e^{-\frac{(w-E)^2}{4\alpha}}dw \\ &\quad +e^{\alpha b^2}\left(H_T(E;ib)-e^{-Eb}\right)-R_1-R_2 \end{aligned} \tag{8.65}$$

where we have introduced the notation of Lemma 1.4. For positive E, α, and b, the equality

$$\frac{1}{2\sqrt{\pi\alpha}}\int_{-\infty}^{\infty}e^{-wb}e^{-\frac{(w-E)^2}{4\alpha}}\,dw = e^{-Eb}e^{\alpha b^2} \tag{8.66}$$

holds; this can be seen e.g. by continuation and evaluation of (1.73) at $t = ib$. Moreover, since $E \geq \gamma$, the first two terms on the right-hand-side of (1.65) can be estimated by

$$\frac{1}{2\sqrt{\pi\alpha}}\int_{-\infty}^{0}e^{-bw}e^{-\frac{(w-E)^2}{4\alpha}}dw \leq \frac{1}{2}e^{-\frac{\gamma^2}{4\alpha}} \tag{8.67}$$

if $2\alpha b \leq \gamma$. Using the bounds established in Lemma 1.4, (1.63) now follows by an application of dominated convergence.

All of our estimates above combine to demonstrate that the right hand side of (1.52) is bounded by

$$\|A\|\,\|B\|\left[e^{-\frac{\gamma^2}{4\alpha}}+\frac{2\,D_a(X,Y)}{\pi\,\|\Phi\|_a\,C_a}\left(e^{2\,\|\Phi\|_a\,C_a\,s}-1\right)+\frac{1}{s\sqrt{\pi\alpha}}e^{-\alpha s^2}\right] \tag{8.68}$$

if α satisfies $\gamma \geq 2\alpha b$. The choice $\alpha = \gamma/2s$ yields:

$$\|A\|\,\|B\|\,e^{-\frac{\gamma s}{2}}\left[1+\sqrt{\frac{2}{\pi\gamma s}}+\frac{2\,D_a(X,Y)}{\pi\,\|\Phi\|_a\,C_a}e^{\left(2\,\|\Phi\|_a\,C_a+\frac{\gamma}{2}\right)s}\right] \tag{8.69}$$

As is demonstrated in (**??**), $D_a(X,Y)$ decays exponentially as $e^{-ad(X,Y)}$. In this case, if we choose s to be the solution of the equation

$$s\,(\,2\,\|\Phi\|_a\,C_a\,+\,\gamma/2\,)\;=\;a\,d(X,Y), \tag{8.70}$$

then we have proven the result. Notice that we have chosen α in terms of s, which is defined independently of b, thus the condition $\gamma \geq 2\alpha b$ will be satisfied for sufficiently small $b > 0$. ☐

In the proof above we used the following lemma. We use the following notation.

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}\,. \tag{8.71}$$

Lemma 8.4. *Fix $\alpha > 0$, $E \in \mathbb{R}$, and $z \in \mathbb{C}^+$. One has that*

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{iEt}\,e^{-\alpha t^2}}{t-z}\,dt \;=\; \frac{1}{2\sqrt{\pi\alpha}}\int_0^{\infty}e^{iwz}\,e^{-\frac{(w-E)^2}{4\alpha}}\,dw. \tag{8.72}$$

Moreover, the convergence is uniform with respect to z in compact subsets of $\mathbb{C}^+$.

Proof. For any $\alpha > 0$, it is easy to check that

$$\frac{1}{2\sqrt{\pi\alpha}}\int_{-\infty}^{\infty}e^{iwt}\,e^{-\frac{(w-E)^2}{4\alpha}}\,dw = e^{iEt}\,e^{-\alpha t^2} \quad \text{for all } t, E \in \mathbb{R}\,. \tag{8.73}$$

Up to appropriate normalizations, this is just the observation that the Fourier transform of a gaussian is a gaussian. Given $z \in \mathbb{C}^+$, it is now clear that for any $T > 0$,

$$\frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{iEt}\,e^{-\alpha t^2}}{t-z}\,dt = \frac{1}{2\sqrt{\pi\alpha}}\int_{-\infty}^{\infty}H_T(w;z)\,e^{-\frac{(w-E)^2}{4\alpha}}\,dw \tag{8.74}$$

where we have denoted by $H_T(w;z)$ the function

$$H_T(w;z) \;=\; \frac{1}{2\pi i}\int_{-T}^{T}\frac{e^{iwt}}{t-z}\,dt \quad \text{for any } T>0 \text{ and } z\in\mathbb{C}^+. \tag{8.75}$$

Rewriting the integral on the right-hand-side of (1.74), we find that

$$\frac{1}{2\sqrt{\pi\alpha}}\int_{-\infty}^{\infty} H_T(w;z)\,e^{-\frac{(w-E)^2}{4\alpha}}\,dw = \frac{1}{2\sqrt{\pi\alpha}}\int_{0}^{\infty} e^{iwz}\,e^{-\frac{(w-E)^2}{4\alpha}}\,dw \,+\, R_1 \,+\, R_2$$

where we have labeled by R_1 and R_2 the integrals

$$\frac{1}{2\sqrt{\pi\alpha}}\int_{0}^{\infty}\left(H_T(w;z)-e^{iwz}\right)e^{-\frac{(w-E)^2}{4\alpha}}\,dw \quad \text{and} \quad \frac{1}{2\sqrt{\pi\alpha}}\int_{-\infty}^{0} H_T(w;z)\,e^{-\frac{(w-E)^2}{4\alpha}}\,dw \tag{8.76}$$

respectively.

We now show that: for fixed $\alpha>0$ and $E\in\mathbb{R}$, the above integrals R_1 and R_2 go to zero as $T\to\infty$. To see this, we inspect the function $H_T(w;z)$. There are two cases to consider.

Let $w>0$, i.e. we estimate R_1. With $z\in\mathbb{C}^+$ fixed, take $T>0$ large enough so that $0<2|z|<T$. In this case, integration about the rectangular contour Γ_T with corners:$-T$, T, $T+iT$, and $-T+iT$ produces:

$$\frac{1}{2\pi i}\int_{\Gamma_T}\frac{e^{iwz'}}{z'-z}\,dz' \;=\; e^{iwz} \tag{8.77}$$

and consequently the bound

$$\left|H_T(w;z)-e^{iwz}\right| \;\leq\; \frac{2}{\pi}\left[\frac{1}{wT}\left(1-e^{-wT}\right) \,+\, e^{-wT}\right]. \tag{8.78}$$

For $w<0$, a similar rectangular contour, now in the lower half plane, can be integrated to yield

$$|H_T(w;z)| \;\leq\; \frac{2}{\pi}\left[\frac{1}{|w|T}\left(1-e^{-|w|T}\right) \,+\, e^{-|w|T}\right]. \tag{8.79}$$

The claimed result, including the statement regarding uniformity, now follows from an application of dominated convergence. □

9. Lie Groups and Lie Algebras

A *Lie Group* G is a group G with a compatible structure of a smooth (real or complex) manifold. For compatibility, it is assumed that the product and inversion mappings are smooth.

Many interesting Lie Groups, and the focus for most of these notes, are matrix Lie Groups which are subgroups of $GL(n,\mathbb{R})$, resp. $GL(n,\mathbb{C})$, here denoting the set of real, resp. complex, $n \times n$ invertible matrices. Two key examples are

$$SO(n) = \{A \in GL(n,\mathbb{R}) : AA^t = \mathbb{1} \text{ and } \det(A) = 1\} \tag{9.1}$$

and

$$SU(n) = \{A \in GL(n,\mathbb{C}) : AA^* = \mathbb{1} \text{ and } \det(A) = 1\} \tag{9.2}$$

both defined for any integer $n \geq 2$.

A Lie Group G is *compact* if G is compact as a manifold. A Lie Group G is *connected* if any two points in G can be linked together by a continuous curve in G.

In general, a *Lie Algebra* g is a vector space g , over a field $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, equipped with a mapping $[\cdot,\cdot] : g \times g \to g$ which satisfies:
i) $[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z]$ for all $x, y, z \in g$ and all $\alpha, \beta \in \mathbb{F}$.
ii) $[x,y] = -[y,x]$ for all $x, y \in g$.
iii) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for all $x, y, z \in g$.

If g is a Lie Algebra, then the mapping $[\cdot,\cdot]$ described above is called the *Lie bracket* associated with g .

If G is a Lie Group, then the Lie Algebra g associated to G is, $T_{\mathbb{1}}(G)$, i.e. the tangent space of G at the identity $\mathbb{1} \in G$. The corresponding Lie bracket, i.e. $[x,y]$ for $x, y \in T_{\mathbb{1}}(G)$, is the standard vector field commutator of the vectors x and y pushed forward by left-multiplication, i.e.

$$[x,y] = [Lx, Ly]_{\mathbb{1}} \quad \text{for all } x, y \in T_{\mathbb{1}}(G)\,. \tag{9.3}$$

It will be important to keep the following examples in mind.

$$SU(2) = \{A \in GL(2,\mathbb{C}) : AA^* = \mathbb{1} \text{ and } \det(A) = 1\} \tag{9.4}$$

One readily checks that

$$SU(2) = \left\{ A = \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} : (z,w) \in \mathbb{C}^2 \text{ and } |z|^2 + |w|^2 = 1 \right\} \tag{9.5}$$

In fact, it is easy to see that any matrix A with form given by (2.5) is in SU(2). Conversely, if $A \in SU(2)$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1} = A^* = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \tag{9.6}$$

and thus A has the form of (2.5). With (2.5), it is clear that SU(2) is equivalent to the unit sphere in $\mathbb{C}^2$. Since the unit sphere in $\mathbb{C}^2$ is equivalent to the real 3-sphere, i.e. the unit sphere in $\mathbb{R}^4$, the same is true for SU(2). In fact, let us define the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.7}$$

Observe that any $A \in SU(2)$ can be written as:

$$A = x_0\mathbb{1} + i\sum_{j=1}^{3} x_j\sigma^j \quad \text{for some real } x_j \text{ satisfying } 1 = \sum_{j=0}^{3} x_j^2\,, \tag{9.8}$$

where we have taken A with the form of (2.5) and written $z = x_0 + ix_3$ and $w = -x_2 - ix_1$. This smooth invertible map between SU(2) and the 3-sphere demonstrates that SU(2) is connected. It is also clearly compact.

Let us now consider SO(3). The group SO(3) is easily seen to be generated by the following three matrices:

(9.9)

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, R_2(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, R_3(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\theta \in [0, 2\pi)$. This can be argued as follows. As a matrix over the complex field, any $A \in SO(3)$ can be diagonalized, since every such A is normal. By orthogonality, any eigenvalue of A must have unit modulus. If all eigenvalues are real, then there are only two choices: all are 1 or 2 are -1. As the entries of A are real valued, any complex eigenvalues must come in conjugate pairs. Since the product of all the eigenvalues is 1, we conclude that A must have an eigenvector corresponding to eigenvalue one. Using the standard basis vectors to extend this normalized vector to an orthonormal basis, we see that any $A \in SO(3)$ is unitarily equivalent to a matrix as in (2.9) above.

Consider the mapping $f : SU(2) \to SO(3)$ given by

$$f(A)_{ij} = \frac{1}{3}\mathrm{Tr}[\sigma^i A \sigma^j A^*] \tag{9.10}$$

One can check that this mapping is indeed into SO(3). In fact, it is onto, $f(-A) = f(A)$, and this mapping is in fact 2 to 1. Moreover, $f(AB) = f(A)f(B)$, i.e. f is a group homomorphism.

Note: we have parametrized the matrices in (2.9) above so that $R_j(0) = \mathbb{1}$ for all $j = 1, 2, 3$. It is easy to see that

$$R_1'(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, R_2'(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, R_3'(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.11}$$

The matrices form a basis for the corresponding Lie Algebra so(3) .

9.1. **Representations of Lie Groups and Lie Algebras.** In this section, we will consider some of the general theory of finite dimensional representations of both Lie Groups and Lie Algebras. Again, most of the focus will be on matrix Lie Groups and their associated Lie Algebras. Throughout this section, we will denote by V a finite dimensional vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By $L(V)$, we will denote the set of all linear transformations from V to V. By $GL(V)$, we will denote the set of all invertible linear transformations from V to V.

Definition 9.1. Let G be a Lie Group and V be a finite dimensional vector space over $\mathbb{K}$. A representation Π of G acting on V is a mapping $\Pi : G \to GL(V)$ which satisfies

$$\Pi(g_1 g_2) = \Pi(g_1)\Pi(g_2) \quad \text{for all } g_1, g_2 \in G\,. \tag{9.12}$$

The dimension of the representation Π is defined by $\dim(\Pi) = \dim(V)$.

It follows immediately from this definition that:

i) $\Pi(\mathbb{1}) = \mathrm{id}$, where id is the identity map on V.

ii) $\Pi(g^{-1}) = (\Pi(g))^{-1}$ for all $g \in G$.

iii) If $\Pi : G \to L(V)$ satisfying $\Pi(\mathbb{1}) = \mathrm{id}$ and (2.12), then Π is a representation of G acting on V.

The definition of representation for a Lie Algebra corresponding to a Lie Group is similar.

Definition 9.2. Let g be a Lie Algebra and V a finite dimensional vector space over $\mathbb{K}$. A representation π of g acting on V is a mapping $\pi : g \to L(V)$ which satisfies

$$\pi([x, y]) = [\pi(x), \pi(y)] \quad \text{for all } x, y \in g\,. \tag{9.13}$$

The dimension of the representation π is defined by $\dim(\pi) = \dim(V)$.

Examples:

1) For any Lie Group G and any vector space V, the trivial representation is the mapping $\Pi(g) = \text{id}$ for all $g \in G$. For g a Lie Algebra, the trivial representation of g is defined by $\pi(x) = 0$ for all $x \in g$.

2) If G is a matrix Lie Group, i.e. a sub-group of $GL(n,\mathbb{R})$, resp. $GL(n,\mathbb{C})$, then with $V = \mathbb{R}^n$, resp. $V = \mathbb{C}^n$, the mapping $\Pi(g) = g$ for all $g \in G$ is called the fundamental representation of G. In this case, one can also define the fundamental representation of the corresponding Lie Algebra, and it is $\pi(x) = x$ for all $x \in g$.

3) Let G be a matrix Lie Group and g the corresponding Lie Algebra. The mapping $\text{Ad} : G \to GL(g)$ defined by

$$\text{Ad}(g)x = gxg^{-1} \quad \text{for all } g \in G \text{ and } x \in g \tag{9.14}$$

is a representation of G on g . This is called the adjoint representation of the Lie Group G. There is an analogous representation of the Lie Algebra g acting on g which is the mapping $\text{ad} : g \to L(g)$ given by

$$(\text{ad}x)(y) = [x, y] \quad \text{for all } x, y \in g. \tag{9.15}$$

Proposition 9.3. *Let G be a matrix Lie Group. If Π is a representation of G acting on V, then there exists a representation π of the corresponding Lie Algebra g , also acting on V, defined via*

$$\pi(x) = \frac{d}{dt}\left(\Pi\left(\exp(tx)\right)\right)|_{t=0} \quad \textit{for all } x \in g\,. \tag{9.16}$$

π is said to be the representation of g induced by Π.

Example:

For any matrix Lie Group G, the adjoint representation Ad, see (2.14), induces the adjoint representation ad, see (2.15). In fact,

$$\frac{d}{dt}\left(\text{Ad}(e^{tx})\right)|_{t=0} = \text{ad}(x) \quad \text{for all } x \in g\,. \tag{9.17}$$

There is a (partial) converse of Proposition 2.3.

Proposition 9.4. *Let G be a matrix Lie Group, and let π be a representation of the corresponding Lie Algebra g acting on V. The mapping*

$$\Pi(g) = e^{\pi(x)} \quad \textit{defined for all } g \in G \textit{ with } g = e^x\,, \tag{9.18}$$

defines a representation Π of G on V.

Since $\mathbb{1} \in G$ satisfies $\mathbb{1} = e^0$, it is clear that the above is well-defined for all x near 0. For this reason, the mapping Π is said to be the representation locally induced by π.

The following is an important definition.

Definition 9.5. Let Π be a representation of a Lie Group G on a finite dimensional vector space V over $\mathbb{C}$. If $\Pi(g)\Pi(g)^* = \text{id}$ for all $g \in G$, then Π is said to be a unitary representation. Similarly, let π be a representation of a Lie Algebra g on a finite dimensional vector space V over $\mathbb{C}$. If $\pi(x)^* = -\pi(x)$ for all $x \in g$, then π is said to be an anti-hermitian representation.

Proposition 9.6. *Let G be a matrix Lie Group and g the corresponding Lie Algebra.*
a) If Π is a unitary representation of G on V, then the induced representation π of g on V is anti-hermitian.
b) If π is an anti-hermitian representation of g on V, then the (locally) induced representation Π of G on V is unitary.

The following definition applies to representations of Lie Groups and Lie Algebras. In general, we will denote such a representation by ρ.

Definition 9.7. Let ρ be a representation of G, which may either be a Lie Group or a Lie Algebra, acting on a finite dimensional vector space V. If $W \subset V$ is a subspace and $\rho(g)w \in W$ for all $g \in G$ and $w \in W$, then W is said to be a ρ-invariant subspace. A representation ρ is said to be reducible if there exists a non-trivial ρ-invariant subspace, i.e. a ρ-invariant subspace $W \subset V$ with $W \neq \{0\}$ and $W \neq V$. If the only ρ-invariant subspaces of V are $\{0\}$ and V, then ρ is said to be irreducible.

Proposition 9.8. *Let G be a matrix Lie Group and $\mathfrak{g}$ the associated Lie Algebra.*
a) If Π is a representation of G acting on V and W is a Π-invariant subspace, then W is also an π-invariant subspace of V, where π is the induced representation of $\mathfrak{g}$ acting on V.
b) If π is a representation of $\mathfrak{g}$ acting on V and W is a π-invariant subspace, then W is also an Π-invariant subspace of V, where Π is the (locally) induced representation of G acting on V.

Definition 9.9. Let ρ be a representation of G, which may either be a Lie Group or a Lie Algebra, acting on a finite dimensional vector space V. ρ is said to be totally reducible if there exists a direct sum decomposition of V into subspaces $\{W_j\}_{j=1}^k$, i.e.,

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k \tag{9.19}$$

with each W_j being a ρ-invariant subspace for which $\rho_j = \rho|_{W_j}$ is irreducible.

Note that if ρ is a totally reducible representation on a finite dimensional vectors space V, then there is a basis of V for which

$$\rho(g) = \begin{pmatrix} \rho_1(g) & 0 & 0 & 0 \\ 0 & \rho_2(g) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \rho_k(g) \end{pmatrix} \tag{9.20}$$

i.e. ρ acts block-diagonally.

Proposition 9.10. *Any finite dimensional unitary representation is totally reducible.*

Proof. Let Π be a unitary representation of a Lie Group G acting on a finite dimensional vector space V over $\mathbb{C}$. If Π is irreducible, we are done. Otherwise, there exists a non-trivial Π-invariant subspace $W \subset V$. We can write $V = W \oplus W^\perp$ and note that if $x \in W^\perp$, $y \in W$, and $g \in G$, then

$$\langle \Pi(g)x, y\rangle = \langle x, \Pi(g)^* y\rangle = \langle x, \Pi(g)^{-1} y\rangle = \langle x, \Pi(g^{-1})y\rangle = 0 \tag{9.21}$$

where the last equality follows since W is Π-invariant. This proves that $W^\perp$ is also a Π-invariant subspace. Iterating this argument to the restrictions of Π to these (finite dimensional) subspaces produces the desired result. □

On equivalent representations:

Definition 9.11. Let ρ_1 be a representation of G, which may either be a Lie Group or a Lie Algebra, acting on finite dimensional vector spaces V_1. For any invertible linear transformation $T : V_1 \to V_2$, then the mapping

$$\rho_2(g) = T\rho_1(g)T^{-1} \tag{9.22}$$

defines a representation of G on V_2. Representations of this type are said to be equivalent representations.

It is easy to check that if Π_1 and Π_2 are equivalent representations, then the corresponding induced representations π_1 and π_2 are equivalent as well. Similarly, if π_1 and π_2 are equivalent, then the locally induced representations Π_1 and Π_2 are also equivalent.

Lemma 9.12. *Let ρ_1 and ρ_2 be two irreducible representations of G acting on vector spaces V_1 and V_2 respectively. If there exists a linear transformation $T : V_1 \to V_2$ for which*

$$T\rho_1(g) = \rho_2(g)T \quad \text{for all } g \in G, \tag{9.23}$$

then either ρ_1 and ρ_2 are equivalent representations or $T = 0$.

Proof. It is easy to check that $\ker(T) \subset V_1$ is a ρ_1-invariant subspace and similarly $\text{ran}(T) \subset V_2$ is a ρ_2-invariant subspace. The claimed result follows. □

Lemma 9.13. *Let ρ be an irreducible representation of G on a finite dimensional vector space V over $\mathbb{C}$. If $T : V \to V$ is linear and*

$$T\rho(g) = \rho(g)T \quad \text{for all } g \in G, \tag{9.24}$$

then $T = \lambda \text{id}$ for some $\lambda \in \mathbb{C}$.

Proof. As a linear operator on a finite dimensional vector space over $\mathbb{C}$, T has at least one eigenvalue $\lambda \in \mathbb{C}$. The non-empty eigenspace U_λ is clearly a ρ-invariant subspace. □

On tensor products. Fix a Lie Group G. For $j = 1, 2$, let Π_j be a representation of G acting on a vector space V_j. Consider the vector space

$$V = V_1 \otimes V_2 \tag{9.25}$$

For any $g \in G$, we define a linear mapping $\Pi(g) : V \to V$ by setting

$$\Pi(g)v_1 \otimes v_2 = (\Pi_1(g)v_1) \otimes (\Pi_2(g)v_2) \ , \tag{9.26}$$

e.g. on a collection of basis vectors, and extending by linearity. The mapping Π is a representation G, it is called the tensor product representation, and it is often denoted by $\Pi = \Pi_1 \otimes \Pi_2$. Similarly, if π_1 and π_2 are representations of g acting on vector spaces V_1 and V_2 respectively, then π defined by setting

$$\pi = \pi_1 \otimes \text{id} + \text{id} \otimes \pi_2 \tag{9.27}$$

is a representation of g acting on $V = V_1 \otimes V_2$. In fact, if Π_1 and Π_2 are representations of a matrix Lie Group acting on V_1 and V_2 respectively, then the induced representation of $\Pi = \Pi_1 \otimes \Pi_2$ acting on $V = V_1 \otimes V_2$ is given by $\pi = \pi_1 \otimes \text{id} + \text{id} \otimes \pi_2$.

9.2. **Irreducible Representations of SU(2).** The Lie Algebra corresponding to $SU(2)$, which we denote by SU(2) , has as basis vectors:

$$T_1 = -\frac{i}{2}\sigma^1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}, \quad T_2 = -\frac{i}{2}\sigma^2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \text{and} \quad T_3 = -\frac{i}{2}\sigma^3 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}. \tag{9.28}$$

Let π be an irreducible representation of SU(2) on a finite dimensional vector space V with $\dim(V) = n$. Define linear mappings

$$J_3 = i\pi(T_3) \quad \text{and} \quad J_\pm = \frac{i}{\sqrt{2}}\left(\pi(T_1) \pm i\pi(T_2)\right) . \tag{9.29}$$

The commutation relations

$$[J_3, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = J_3 \tag{9.30}$$

are easily checked.

Since V is a complex vector space, it is clear that J_3 has at least one eigenvalue $\lambda \in \mathbb{C}$. Let us denote a corresponding (non-zero) eigenvector by ϕ. One readily calculates that

$$J_3(J_\pm)^k\phi = (\lambda \pm k)(J_\pm)^k\phi \quad \text{for any integer } k \geq 0. \tag{9.31}$$

The mappings $J_\pm$ are called raising and lowering operators. The eigenvalues of J_3 are called weights. The above calculation shows that either $(J_\pm)^k\phi = 0$ or it is an eigenvector of J_3 with eigenvalue $\lambda \pm k$.

Consider the sequence of vectors $\{(J_+)^k\phi\}_{k=0}^\infty \subset V$. The non-zero vectors in this sequence are eigenvectors of J_3 corresponding to distinct eigenvalues, and therefore, linearly independent. Since the vector space V is finite dimensional, there must exist a number $j \geq 0$ for which $(J_+)^j\phi \neq 0$ but $(J_+)^{j+1}\phi = 0$. The corresponding eigenvalue of J_3, which we denote by λ_j, is called the highest weight of the representation π. It is convenient to relabel $\psi = (J_+)^j\phi \neq 0$.

Mimicking the above calculation, we find that

$$J_3(J_-)^k\psi = (\lambda_j - k)(J_-)^k\psi \quad \text{for all integers } k \geq 0. \tag{9.32}$$

Again, these are either 0 or linearly independent eigenvectors of J_3. Using again that V is finite dimensional, there is a number $m \geq 0$ such that $0 \neq (J_-)^m\psi$, but $(J_-)^{m+1}\psi = 0$. For $k = 0, \cdots, m$, denote by $\psi_k = (J_-)^k\psi$. Note that $\psi_0 = \psi$, and for all $0 \leq k \leq m$, $\psi_k \neq 0$ and $J_3\psi_k = (\lambda_j - k)\psi_k$.

We will now prove that $2\lambda_j = m$. Recall the integer $m \geq 0$. Consider the case $m = 0$. In this case, there is a non-zero vector $\psi = \psi_0$ that satisfies $J_+\psi = 0 = J_-\psi$. As a result,

$$\lambda_j\psi = J_3\psi = [J_+, J_-]\psi = 0 \quad \Rightarrow \quad 2\lambda_j = 0 = m. \tag{9.33}$$

Now, if $m \geq 1$, then it is easy to see that for $1 \leq k \leq m$

$$\begin{aligned}
J_+\psi_k = J_+(J_-)^k\psi_0 &= \left([J_+, J_-] + J_-J_+\right)(J_-)^{k-1}\psi_0 \\
&= J_3(J_-)^{k-1}\psi_0 + J_-\left([J_+, J_-] + J_-J_+\right)(J_-)^{k-2}\psi_0 \\
&= \sum_{\ell=0}^{k-1}(J_-)^\ell J_3(J_-)^{k-1-\ell}\psi_0 \\
&= k\left(\lambda_j - \frac{1}{2}(k-1)\right)\psi_{k-1} \qquad (9.34)
\end{aligned}$$

where we have used (2.32) and that, by construction, $J_+\psi_0 = J_+\psi = 0$. Since we also have that $J_-\psi_m = 0$, we find that

$$\begin{aligned}
0 = J_+J_-\psi_m = \left([J_+, J_-] + J_-J_+\right)\psi_m &= (\lambda_j - m)\psi_m + J_-J_+\psi_m \\
&= \left[(\lambda_j - m) + m\left(\lambda_j - \frac{1}{2}(m-1)\right)\right]\psi_m \\
&= \frac{1}{2}(m+1)(2\lambda_j - m)\psi_m \qquad (9.35)
\end{aligned}$$

Since $\psi_m \neq 0$, we have then that $2\lambda_j = m$ as claimed.

We now claim that the vectors $\{\psi_k\}_{k=0}^m$ form a basis of V and that the highest weight vector $\psi_0 = \psi$ is unique up to normalization. Consider the subspace $V' \subset V$ spanned by these vectors. This is easily checked to be a non-empty π-invariant subspace, and hence $V' = V$ since π is irreducible. As a consequence, we have also proven that J_3 is diagonalizable on V and that each eigenspace is one-dimensional. Moreover, $n = \dim(V) = m + 1 = 2\lambda_j + 1$.

To see that the highest weight vector is unique, suppose that φ is an eigenvector of J_3 with $J_3\varphi = \mu\varphi$ and $J_+\varphi = 0$. As the vectors $\{\psi_k\}_{k=0}^m$ form a basis of V, it is clear that

$$\varphi = \sum_{k=0}^m c_k\psi_k \tag{9.36}$$

We also showed above that the eigenspaces of J_3 are one-dimensional. Thus there is some $0 \leq k_0 \leq m$ for which $\varphi = c_{k_0}\psi_{k_0}$. If $k_0 = 0$, we are done. Otherwise, we have that

$$\mu\varphi = J_3\varphi = c_{k_0}(\lambda_j - k_0)\psi_{k_0} = (\lambda_j - k_0)\varphi \quad \Rightarrow \quad \mu = \lambda_j - k_0 \tag{9.37}$$

We also know that

$$0 = J_+\varphi = c_{k_0} J_+\psi_{k_0} = c_{k_0} k_0 \left(\lambda_j - \frac{1}{2}(k_0 - 1)\right)\psi_{k_0-1} \quad \Rightarrow \quad \lambda_j = \frac{1}{2}(k_0 - 1) \tag{9.38}$$

But we know that $k_0 \leq m = 2\lambda_j$. Thus $2\lambda_j = k_0 - 1 \leq 2\lambda_j - 1$ an obvious contradiction. We conclude $k_0 = 0$ and we are done.

9.3. **Tensor products of representations.** Let π_1 and π_2 be two irreducible representations of $SU(2)$ on vector spaces $V^{(1)}$ and $V^{(2)}$. We will consider the representation

$$\pi = \pi_1 \otimes \mathbb{1} + \mathbb{1} \otimes \pi_2 \tag{9.39}$$

acting on $V = V^{(1)} \otimes V^{(2)}$. Our goal is to decompose V into irreducible representations of π, i.e. write V as a direct sum of π-invariant subspaces with the property that on each π acts as an irreducible representation.

We begin with some notation. As before, for $k = 1, 2$, set

$$J_3^{(k)} = i\pi_k(T_3) \quad \text{and} \quad J_\pm^{(k)} \frac{i}{\sqrt{2}}\left(\pi_k(T_1) \pm i\pi_k(T_2)\right) \tag{9.40}$$

On V we define

$$J_3 = J_3^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J_3^{(2)} \quad \text{and} \quad J_\pm = J_\pm^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J_\pm^{(2)} \tag{9.41}$$

The commutation relations

$$[J_3, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = J_3 \tag{9.42}$$

are easily checked.

Let us now suppose that $\dim(V^{(k)}) = 2j_k + 1$ for $k = 1, 2$. Denote by $\{\psi_m^{(k)}\}$ the sequences of eigenvectors of $J_3^{(k)}$ constructed previously, i.e.

$$J_3^{(k)}\psi_m^{(k)} = m\psi_m^{(k)} \quad \text{for all } -j_k \leq m \leq j_k\,. \tag{9.43}$$

It is clear that vectors of the form

$$\Psi_{m,n} = \psi_m^{(1)} \otimes \psi_n^{(2)} \tag{9.44}$$

form a basis of V and moreover, one readily checks that

$$J_3\Psi_{m,n} = (m+n)\Psi_{m,n} \tag{9.45}$$

Let us denote by E_λ the eigen-subspace of V corresponding to the operator J_3 and the eigenvalue λ. By construction, it is clear that $\dim(E_{j_1+j_2}) = 1$. In fact, one also sees that

$$\dim(E_{j_1+j_2-k}) = k+1 \quad \text{for all } 0 \leq k \leq j_1 + j_2 - |j_1 - j_2| \tag{9.46}$$

This is because there are $k+1$ ways to write the vectors above in such a way that the sum of their eigenvalues is this number . . . The collection of these eigenspaces does not exhaust the whole of V, but they will be sufficient for our purposes.

We find this direct sum decomposition algorithmically. We start with the highest weight vector, i.e. Ψ_{j_1,j_2}. This vector is the unique vector, up to normalization, in the eigenspace of J_3 corresponding to $j_1 + j_2$. Iteratively applying J_- to this vector, we produce $2(j_1 + j_2) + 1$ vectors in V that are eigenvectors of J_3 with eigenvalues m with $-(j_1 + j_2) \leq m \leq j_1 + j_2$. The subspace spanned by these orthogonal vectors, which we will denote by $V_{j_1+j_2}$, is a π-invariant subspace of V on which π acts an irreducible representation. Take $V' = V_{j_1+j_2}^{\perp}$.

Now consider $\lambda_1 = j_1 + j_2 - 1$. If $1 \leq j_1 + j_2 - |j_1 - j_2|$, then the dimension of the eigenspace of J_3 corresponding to λ_1 is 2, using (2.46). One of these vectors was constructed in the previous argument and hence it lies in $V_{j_1+j_2}$. Since the dimension of the eigenspace is 2, we are guaranteed that there exists a linearly independent vector in V' which is also an eigenvector of J_3 with eigenvalue λ_1. Using it and iteratively applying J_-, we find $2(j_1 + j_2 - 1) + 1$ orthogonal vectors in V' - each eigenvectors of J_3 with prescribed eigenvalues. The subspace spanned by these vectors, which we will denote by $V_{j_1+j_2-1}$ is π-invariant and π acts on this subspace as an irreducible representation. Relabel $V' = (V_{j_1+j_2} \oplus V_{j_1+j_2-1})^{\perp}$.

Now, by way of induction, let $\lambda_k = j_1 + j_2 - k$ and assume that we have continued as above k times. If $k \leq j_1 + j_2 - |j_1 - j_2|$, then the eigenspace associated to J_3 and eigenvalue λ_k is $k+1$ dimensional by (2.46). By way of induction, we have already selected k linearly independent vectors in this subspace. In this step, we select the remaining vector. We apply J_- iteratively and obtain $2\lambda_k + 1$ orthogonal vectors which span the π-invariant subspace V_{λ_k}.

This procedure cannot be continued beyond $k = j_1 + j_2 - |j_1 - j_2|$. At that point, we have a subspace of V with dimension

$$\begin{aligned} \sum_{k=|j_1-j_2|}^{j_1+j_2} \dim(V_k) = \sum_{k=|j_1-j_2|}^{j_1+j_2} 2k+1 &= \sum_{\ell=0}^{j_1+j_2-|j_1-j_2|} (2(|j_1-j_2|+\ell)+1) \\ &= (2j_1+1)(2j_2+1) = \dim(V) \end{aligned} \tag{9.47}$$

and therefore, we have decomposed all of V.

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