An Introduction to Quantum Spin Systems[1]
Notes for MA5020 (John von Neumann Guest Lectures)

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Contents

4. General Framework for Finite and Infinite Quantum Spin Systems

The general framework introduced in this chapter will allow us to consider infinite quantum spin systems as C^*-dynamical systems. In particular we will construct the dynamics for infinite quantum spin systems as a strongly continuous one-parameter group of automorphism of the algebra of quasi-local observables. We start by studying the dynamics of finite quantum spin systems.

4.1. **The Dynamics of Finite Systems.** Let Λ be a finite set. For each $x \in \Lambda$ we have a quantum system described by a finite-dimensional Hilbert space of dimension $d_x \geq 2$. These are the 'spins' that form the spin system. The Hilbert space for the finite spin system is then

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^{d_x}. \tag{4.1}$$

The algebra of observables of the systems is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} M_{d_x}. \tag{4.2}$$

Due to the tensor product structure, for any $\Lambda_1 \subset \Lambda$, the collection of observables $\mathcal{A}_{\Lambda_1}$ may be regarded as a subset of the observables in $\mathcal{A}_\Lambda$ by identifying $A \in \mathcal{A}_{\Lambda_1}$ with $A \otimes \mathbb{1}_{\Lambda\setminus\Lambda_1} \in \mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda\setminus\Lambda_1} = \mathcal{A}_\Lambda$. With this in mind, we will consider $\mathcal{A}_{\Lambda_1}$ as a subalgebra of $\mathcal{A}_\Lambda$.

An interaction Φ is a mapping $\Phi : \mathcal{P}(\Lambda) \to \mathcal{A}_\Lambda$ (where $\mathcal{P}(\Lambda)$ denotes the set of all subsets of Λ) with the property that: For each $X \in \mathcal{P}(\Lambda)$, $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X)^* = \Phi(X)$. For any $Z \subset \Lambda$, the Hamiltonian corresponding to Φ in the volume Z is given by

$$H_Z = \sum_{X\in\mathcal{P}(Z)} \Phi(X) \tag{4.3}$$

Example An interaction, Φ_H, for the Heisenberg spin chain on an interval $[a, b] \subset \mathbb{Z}$ is given by

$$\Phi_H(X) = \begin{cases} -J\vec{\sigma}_j\vec{\sigma}_{j+1} & \text{if } X = \{j, j+1\} \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

and therefore we have

$$H_{[a,b]} = \sum_{X\in\mathcal{P}([a,b])} \Phi_H(X) = -J\sum_{j=a}^{b-1} \vec{\sigma}_j\vec{\sigma}_{j+1} \tag{4.5}$$

as in (2.55) above.

The Heisenberg dynamics, which we will denote by τ_t^Λ, generated by the Hamiltonian H_Λ (corresponding to the interaction Φ) is an automorphism of the algebra $\mathcal{A}_\Lambda$ defined as follows:

$$\tau_t^\Lambda(A) = U_\Lambda^*(t)AU_\Lambda(t), \quad \text{for all } A \in \mathcal{A}_\Lambda, \tag{4.6}$$

where $U_\Lambda(t)$ is the unitary operator

$$U_\Lambda(t) = e^{-itH_\Lambda} \in \mathcal{A}_\Lambda. \tag{4.7}$$

The automorphisms τ_t^Λ provide the solutions to the Heisenberg equation for the time-evolution of observables:

$$\frac{d}{dt}\tau_t^\Lambda(A) = i[H_\Lambda, \tau_t^\Lambda(A)]. \tag{4.8}$$

Often, we consider finite volume subsystems of some infinite system of spins labeled by a countable set Γ. A common situation is a spin systems defined on the lattice $\mathbb{Z}^\nu$.

A typical model will be defined by specifying a global interaction Φ which is a mapping

$$\Phi : \mathcal{P}_0(\mathbb{Z}^\nu) \to \bigcup_{n\geq 1} \mathcal{A}_{\Lambda_n} \tag{4.9}$$

with $\mathcal{P}_0(\mathbb{Z}^\nu)$ being the set of *finite* subsets of $\mathbb{Z}^\nu$ and the union of the observable algebras is defined inductively using that $\mathcal{A}_{\Lambda_n} \subset \mathcal{A}_{\Lambda_{n+1}}$. (Under the additional assumption that the sequence is *exhaustive*, i.e., $\cup_n \Lambda_n = \mathbb{Z}^\nu$, this union is independent of the chosen sequence.) The same conditions on the interaction, i.e., $\Phi(X)^* = \Phi(X) \in \mathcal{A}_X$ apply. We often investigate properties of finite-volume Hamiltonians corresponding to this fixed interaction:

$$H_n = H_{\Lambda_n} = \sum_{X \in \mathcal{P}(\Lambda_n)} \Phi(X) \tag{4.10}$$

For the dynamics, it is clear that if $\Lambda_1 \subset \Lambda$, then

$$\tau_t^{\Lambda_1}(A) \in \mathcal{A}_{\Lambda_1} \quad \text{for any } A \in \mathcal{A}_{\Lambda_1} \text{ and all } t \in \mathbb{R}. \tag{4.11}$$

However, it is generally the case that

$$\tau_t^{\Lambda}(A) \notin \mathcal{A}_{\Lambda_1} \quad \text{for any } A \in \mathcal{A}_{\Lambda_1} \text{ and } t \neq 0. \tag{4.12}$$

Let's examine this further in the context of a one-dimensional systems with nearest neighbor interactions, such as the one-dimensional Heisenberg model. Let H_N denote the Hamiltonian for such a system on $[-N, N] \subset \mathbb{Z}$:

$$H_N = \sum_{j=-N}^{N-1} h_{j,j+1} \quad \text{with} \quad h_{j,j+1} = -J\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} \in \mathcal{A}_{\{j,j+1\}} \tag{4.13}$$

The corresponding Heisenberg dynamics, i.e. $\tau_t^N(\cdot)$, can be defined by the series for the exponential of its generator $i[H_N, \cdot]$:

$$\tau_t^N(A) = e^{it[H_N,\cdot]}(A) = A + it[H_N, A] + \frac{(it)^2}{2!}[H_N, [H_N, A]] + \cdots, \quad \text{for any } A \in \mathcal{A}_{[-N,N]}. \tag{4.14}$$

To gain some insight in the structure of the dynamics, consider $A \in \mathcal{A}_{\{0\}}$ i.e. an observable that acts non-trivially only at the origin. Then, using the local form of the Hamiltonian, i.e. (4.13), and the fact that observables with spatially disjoint support commute, we find that the first order term is

$$[H_N, A] = ([h_{-1,0}, A] + [h_{0,1}, A]) \in \mathcal{A}_{\{-1,0,1\}}. \tag{4.15}$$

A similar calculation shows that

$$[H_N, [H_N, A]] \in \mathcal{A}_{\{-2,-1,0,1,2\}} \tag{4.16}$$

and in for general $n \geq 0$,

$$([H_N, \cdot])^n(A) \in \mathcal{A}_{[-\min(n,N), \min(n,N)]}. \tag{4.17}$$

As a consequence, if we take $B \in \mathcal{A}_{\{x\}}$ for some $x \in [-N, N]$, one readily sees that

$$[\tau_t^N(A), B] = \mathcal{O}(|t|^{|x|}), \tag{4.18}$$

suggesting that, for $A \in \mathcal{A}_{\{0\}}$, the commutator of $\tau_t^N(A)$ with $B \in \mathcal{A}_{\{x\}}$, is small for t small and x large. One observes, however, that direct analysis of the series expansion does not look appealing due to the fast growth in n of the number of terms that contribute at order n. In any case, we are interested in an explicit estimate for the norm of commutators of this type. The following Lemma shows how such estimates could be used.

Lemma 4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complex Hilbert spaces. Suppose that, for $\epsilon > 0$, $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies*

$$\|[A, \mathbb{1} \otimes B]\| \leq \epsilon \|B\|, \tag{4.19}$$

for all $B \in \mathcal{B}(\mathcal{H}_2)$. Then there exists $A' \in \mathcal{B}(\mathcal{H}_1)$, $\|A'\| \leq \|A\|$, such that

$$\|A' \otimes \mathbb{1} - A\| \leq \epsilon \tag{4.20}$$

So, if $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ has a small commutator with all $B \in \mathcal{B}(\mathcal{H}_2)$, then A is well-approximated by observable in $\mathcal{B}(\mathcal{H}_1)$, i.e., one with support in the complement of the support of the B's.

If $\dim(\mathcal{H}_2) < \infty$, the local operator in the statement of the lemma can be taken to be

$$A' = \frac{1}{\dim(\mathcal{H}_2)} \mathrm{Tr}_2 A, \tag{4.21}$$

where Tr_2 denotes the partial trace over $\mathcal{H}_2$.

Proof of Lemma 4.1 in the finite-dimensional case: If $\dim \mathcal{H}_2 < \infty$, a simple application of Schur's Lemma shows that, for $B \in \mathcal{B}(\mathcal{H}_2)$,

$$\frac{1}{\dim(\mathcal{H}_2)} \mathrm{Tr} B = \int_{U(\mathcal{H}_2)} U^* B U \, dU, \tag{4.22}$$

where $U(\mathcal{H}_2)$ is the unitary group and dU is the normalized Haar-measure on $U(\mathcal{H}_2)$. From this expression, we see that A' as defined in (4.21) can be expressed by

$$A' \otimes \mathbb{1} = \int_{U(\mathcal{H}_2)} (\mathbb{1} \otimes U^*) A (\mathbb{1} \otimes U) \, dU \tag{4.23}$$

Then,

$$A' \otimes \mathbb{1} - A = \int_{U(\mathcal{H}_2)} \{(\mathbb{1} \otimes U^*) A (\mathbb{1} \otimes U) - (\mathbb{1} \otimes U^*)(\mathbb{1} \otimes U) A\} \, dU, \tag{4.24}$$

and so

$$\|A' \otimes \mathbb{1} - A\| \leq \int_{U(\mathcal{H}_2)} \|(\mathbb{1} \otimes U^*)[A, (\mathbb{1} \otimes U)]\| \, dU \leq \epsilon \tag{4.25}$$

since $\|U\| \leq 1$. This completes the proof for the case of finite-dimensional $\mathcal{H}_2$. For a proof in the case of arbitrary Hilbert spaces and further discussion see [12]. □

In our analysis of commutator bounds we will use solutions of Schrödinger equations with time-dependent Hamiltonians, or so-called non-autonomous quantum systems. This is of course an interesting subject in its own right. Here, we limit ourselves to the simplest situation: that of a norm-continuous function $t \mapsto H(t) \in \mathcal{B}(\mathcal{H})$. We are interested in the initial value problem

$$\begin{aligned} \frac{d}{dt}\psi(t) &= -iH(t)\psi(t) \\ \psi(0) &= \psi_0 \in \mathcal{H}. \end{aligned} \tag{4.26}$$

Existence and uniqueness of the solutions follows from standard results for differential equantions. The following construction shows that the solution can be expressed in terms of a family of unitary operators on $\mathcal{H}$, which, for reasons that will become clear later, we will denote by $U(t,0)$. $U(t,0)$ is given by the following absolutely convergent series, called the Dyson series or sometimes 'time-ordered exponential':

$$U(t,0) = \mathbb{1} + \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, H(t_1) H(t_2) \cdots H(t_n). \tag{4.27}$$

It is straightforward to verify that

$$\frac{d}{dt} U(t,0) = -iH(t)U(t,0), \quad U(0,0) = \mathbb{1}, \tag{4.28}$$

which immediately implies that $\psi(t) = U(t,0)\psi(0)$ solved (4.26), for all $\psi_0 \in \mathcal{H}$. By taking adjoints of both sides of (4.28) we obtain a similar equation for $U(t,0)^*$:

$$\frac{d}{dt} U(t,0)^* = iU(t,0)^* H(t), \quad U(0,0)^* = \mathbb{1}, \tag{4.29}$$

Using (4.28) and (4.29) we find that the derivative of $U(t,0)^*U(t,0)$ vanishes for all $t \in \mathbb{R}$, and we conclude that $U(t,0)$ is unitary as claimed. Therefore, for $s,t \in \mathbb{R}$, we can define a unitary $U(t,s) = U(t,0)U(s,0)^*$. It is easy to verify that $U(t,s)$ satisfies $U(t,s)^* = U(t,s)^{-1} = U(s,t)$ and the cocycle property: $U(t,s)U(s,r) = U(t,r)$, for $r,s,t \in \mathbb{R}$.

The Heisenberg dynamics of observables can then be given in terms of a co-cycle of automorphisms $\tau_{t,s}$ defined by

$$\tau_{t,s}(A) = U(t,s)^* A U(t,s), A \in \mathcal{B}(\mathcal{H}). \tag{4.30}$$

The automorphisms satisfy the equation

$$\frac{d}{dt}\tau_{t,0}(A) = i\tau_{t,0}([H(t),A]) = i[\tau_{t,0}(H(t)),\tau_{t,0}(A)]). \tag{4.31}$$

4.2. **Infinite Systems.** We already indicated that one is often interested in families of finite systems defined on finite subsets Λ of an infinite set Γ, with an interaction Φ defined on $\mathcal{P}_0(\Gamma)$, the finite subsets of Γ. This will be the starting point for the definition of infinite quantum spin systems.

Let (Γ, d) be a countable metric space. We will impose certain regularity conditions on (Γ, d). An example to keep in mind is $\mathbb{Z}^\nu$ with the usual graph (i.e., the ℓ^1) distance. To each $x \in \Gamma$, we associate a finite-dimensional, single-site Hilbert space of states $\mathcal{H}_x = \mathbb{C}^{d_x}$. As before, the algebra of observables at the site x will be denoted by $\mathcal{A}_x = \mathcal{B}(\mathcal{H}_x) = M_{d_x}$. For any finite volume $\Lambda \subset \Gamma$, we then have

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x \quad \text{and} \quad \mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_x \tag{4.32}$$

As we have seen, if $\Lambda_0 \subset \Lambda$ are two finite subsets of Γ, then $\mathcal{A}_{\Lambda_0} \subset \mathcal{A}_\Lambda$. It therefore makes sense to consider the union over all finite subsets of Γ:

$$\mathcal{A}_\Gamma^{\text{loc}} = \bigcup_{\Lambda\subset\Gamma} \mathcal{A}_\Lambda. \tag{4.33}$$

The C^*-algebra of all quasi-local observables is the norm completion of $\mathcal{A}_\Gamma^{\text{loc}}$:

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\text{loc}}}^{\|\cdot\|}. \tag{4.34}$$

An *interaction* Φ is a map from the finite subsets of Γ to $\mathcal{A}_\Gamma^{\text{loc}}$, $\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma^{\text{loc}}$, that satisfies

$$\Phi(X)^* = \Phi(X) \in \mathcal{A}_X \quad \text{for each } X \in \mathcal{P}_0(\Gamma). \tag{4.35}$$

The Heisenberg dynamics associated to this interaction is then defined for any finite $\Lambda \subset \Gamma$ in terms of the self-adjoint finite-volume Hamiltonian

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X), \tag{4.36}$$

For each $\Lambda \in \mathcal{P}_0(\Gamma)$, the finite-volume dynamics is given by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \quad \text{for any } A \in \mathcal{A}_\Lambda \text{ and } t \in \mathbb{R}. \tag{4.37}$$

So far, we have a family, labeled by $\Lambda \in \mathcal{P}_0(\Gamma)$, of one-parameter groups of $*$-automorphisms on $\mathcal{A}_\Lambda$. We are interested in a framework where it makes sense to consider infinite systems describing bulk matter. Note that one can regard the τ_t^Λ as automorphisms defined on $\mathcal{A}_{\Lambda'}$, with $\Lambda \subset \Lambda'$, $\mathcal{A}_\Gamma^{\text{loc}}$, or $\mathcal{A}_\Gamma$, for which $\mathcal{A}_\Lambda$ is an invariant subspace. Therefore, it makes sense to consider the convergence of τ_t^Λ as automorphisms on $\mathcal{A}_\Gamma$ as $\Lambda \nearrow \Gamma$.

A convenient way to express sufficient conditions for the existence of the infinite-volume limit of the dynamics (and other quantities), is by means of a function $F : [0,\infty) \to (0,\infty)$, which we will refer to as an *F-function* if it satisfies the following properties:

i. *Non-increasiing:* for $0 \le r \le s$, we have $F(r) \ge F(s)$;

ii. *Uniform integrability:*

$$\|F\| = \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) < \infty \tag{4.38}$$

iii. *Convolution condition:* There exists $C_F < \infty$ such that for any $x, y \in \Gamma$

$$\sum_{z\in\Gamma} F(d(x,z))F(d(z,y)) \le C_F F(d(x,y)) \tag{4.39}$$

For example, if $\Gamma = \mathbb{Z}^\nu$, then for any $\epsilon > 0$, one can take

$$F(r) = (1+r)^{-(\nu+\epsilon)} \tag{4.40}$$

which is clearly, uniformly integrable. Moreover, it is easy to check that the convolution property holds with

$$C_F = 2^{\nu+\epsilon}\|F\| \tag{4.41}$$

works in (4.39).

It is also convenient to observe that if F is an F-function on (Γ, d) - as described above, then for any $a \ge 0$,

$$F_a(r) = e^{-ar}F(r) \tag{4.42}$$

also satisfies the required properties required i-iii) above (i.e. F_a is also an F-function on (Γ, d)) with $\|F_a\| \le \|F\|$ and $C_{F_a} \le C_F$.

In terms of any F-function on (Γ, d), we can define a Banach space of interactions Φ with the norm

$$\|\Phi\|_F = \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{X\subset\Gamma:\\ x,y\in X}} \|\Phi(X)\|. \tag{4.43}$$

Then, $\mathcal{B}_F(\Gamma) = \{\Phi \mid \Phi \text{ is an interaction s.t. } \|\Phi\|_F < \infty\}$. This norm $\|\cdot\|_F$, often referred to as an F-norm, expresses the decay of the interaction strength at long distances: for each pair of points $x, y \in \Gamma$, the sum over all interaction terms which involve this pair must decay faster than F, in the sense that for any $x, y \in \Gamma$, we have

$$\sum_{\substack{X\subset\Gamma:\\ x,y\in X}} \|\Phi(X)\| \le \|\Phi\|_F F(d(x,y)). \tag{4.44}$$

A commonly used bound for the total interaction energy per spin is

$$|||\Phi|||_0 = \sup_{x\in\Gamma} \sum_{\substack{X\in\mathcal{P}_0(\Gamma)\\ x\in X}} \frac{1}{|X|}\|\Phi(X)\|. \tag{4.45}$$

and is an easy exercise to show $|||\Phi|||_0 \le \|F\|\|\Phi\|_F$. We then also have the frequently used bound

$$\sup_{x\in\Gamma} \sum_{\substack{X\in\mathcal{P}_0(\Gamma)\\ x\in X}} \|\Phi(X)\| \le \|F\|\|\Phi\|_F. \tag{4.46}$$

5. Lieb-Robinson Bounds

We will now state and prove a version of the quasi-locality estimates known as Lieb-Robinson bounds. Lieb-Robinson bounds can be expressed a number of different forms, and the precise manner typically depends on the application one has in mind. Often one is considering a dynamics generated by nearest neighbor interactions. In this case, it seems intuitively clear that the spread of the interactions through the system should depend on the surface area of the support of a local observable, not its volume.

Let $\Lambda \in \mathcal{P}_0(\Gamma)$. For any $X \subset \Lambda$, we will denote the surface of X in Λ by

$$S_\Lambda(X) \,=\, \{Z \subset \Lambda \,:\, Z \cap X \neq \emptyset \text{ and } Z \cap (\Lambda \setminus X) \neq \emptyset\} \tag{5.1}$$

and set $S(X) = S_\Gamma(X)$ for brevity. The Φ-boundary of a set $X \in \mathcal{P}_0(\Gamma)$ is then defined to be

$$\partial_\Phi X \,=\, \{x \in X \,:\, \exists Z \in S(X) \text{ with } x \in Z \text{ and } \Phi(Z) \neq 0\,\}\,. \tag{5.2}$$

It is clear that for general Φ, $\partial_\Phi X = X$, but if Φ is finite range and X is sufficiently large, we have that $\partial_\Phi X$ is a proper subset of X.

A Lieb-Robinson bound may be stated as follows.

Theorem 5.1 (Lieb-Robinson Bound [7, 10, 11, 13])**.** *Let $\Phi \in \mathcal{B}_F(\Gamma)$, $X, Y, \Lambda \in \mathcal{P}_0(\Gamma)$, such that $X, \cap Y = \emptyset$. Then, for all $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, we have the estimate*

$$\left\|[\tau_t^\Lambda(A), B]\right\| \,\leq\, \frac{2\,\|A\|\,\|B\|}{C_F}\,\left(e^{2\,\|\Phi\|_F\,C_F\,|t|} - 1\right)\,D(X, Y), \tag{5.3}$$

holds for all $t \in \mathbb{R}$. Here the quantity $D(X, Y)$ is given by

$$D(X, Y) = \min\left\{\sum_{x\in X}\sum_{y\in\partial_\Phi Y} F\left(d(x, y)\right), \sum_{x\in\partial_\Phi X}\sum_{y\in Y} F\left(d(x, y)\right)\right\}. \tag{5.4}$$

Before we prove this bound, a number of comments are useful in interpreting this theorem.

First, one always has the trivial bound $\|[\tau_t^\Lambda(A), B]\| \leq 2\|A\|\|B\|$. This trivial estimate is usually better when $|t|$ is large and also holds when $X \cap Y \neq \emptyset$.

Next, if Φ is exponentially decaying, i.e. there is $a > 0$ for which $\Phi \in \mathcal{B}_{F_a}(\Gamma)$ with $F_a(r) = e^{-ar}F(r)$, then

$$\begin{aligned} D(X, Y) \;&\leq\; \min\left\{\sum_{x\in X}\sum_{y\in\partial_\Phi Y} F\left(d(x, y)\right), \sum_{x\in\partial_\Phi X}\sum_{y\in Y} F\left(d(x, y)\right)\right\} e^{-ad(X,Y)} \\ &\leq\; \min\left\{|\partial_\Phi X|, |\partial_\Phi Y|\right\}\|F\|e^{-ad(X,Y)} \end{aligned} \tag{5.5}$$

In this case, the bound (5.3) implies

$$\left\|[\tau_t^\Lambda(A), B]\right\| \,\leq\, \frac{2\,\|A\|\,\|B\|\,\|F\|}{C_a}\,\min\left\{|\partial_\Phi X|\,, |\partial_\Phi Y|\right\}\,e^{-a\left[d(X,Y) - \frac{2\|\Phi\|_a C_a}{a}|t|\right]}, \tag{5.6}$$

If Φ is finite range on $\mathbb{Z}^\nu$, then $\Phi \in \mathcal{B}_{F_a}(\mathbb{Z}^\nu)$ for all $a > 0$. For $\Phi \in \mathcal{B}_{F_a}$, with $a > 0$, (5.6) can be interpreted as a bound on the velocity of propagation given by

$$v_{\Phi,a} = \frac{2\|\Phi\|_a C_a}{a}. \tag{5.7}$$

It is also important to observe that for fixed local observables A and B, the bounds above, (5.3) and similarly (5.6) if applicable, are independent of the volume $\Lambda \in \mathcal{P}_0(\Gamma)$. This will be key in our proof of the existence of the infinite-volume dynamics.

Finally, we note that the bound above only depends on the minimal cardinality of Φ-boundaries. Hence, one may still obtain useful estimates even in cases where one of the corresponding observables has support growing with the volume Λ.

In the proof of Theorem 5.1 we will use the lemma below, which provide a simple estimate for the growth of solutions of a class of differential equations in a Banach space. In this lemma the derivative is to be interpreted as a limit in the Banach space norm and the integrals in the proof may be interpreted as Riemann or Bochner integrals.

Let X be a Banach space and let I be a finite or infinite interval $\subset \mathbb{R}$. Suppose $A : I \to \mathcal{B}(X)$ be a continuous function with values in the bounded linear operators on X considered with the operator norm, and denote by $x(t)$ the solution of the differential equation

$$\partial_t x(t) \;=\; A(t)\, x(t) \tag{5.8}$$

with initial condition $x(t_0) = x_0 \in X$. We say that the family of operators $A(t)$ is *norm-preserving* if for every $x_0 \in X$, the mapping $\gamma_t : X \to X$ which associates $x_0 \to x(t)$, i.e., $\gamma_t(x_0) = x(t)$, satisfies

$$\|\, \gamma_t(x_0)\, \| \;=\; \|\, x_0\, \| \quad \text{for all } t \in I. \tag{5.9}$$

Some obvious examples are the case where X is a Hilbert space and $A(t)$ is anti-hermitian for each t, or when X is a $*$-algebra of operators on a Hilbert space with a spectral norm and, for each t, $A(t)$ is a derivation commuting with the $*$-operation.

Lemma 5.2. *Let $A(t)$, for $t \in I \subset \mathbb{R}$, be a family of norm preserving operators on a Banach space X. For any continuous function $b : I \to X$, the solution of*

$$\partial_t y(t) \;=\; A(t) y(t) \,+\, b(t), \tag{5.10}$$

with boundary condition $y(t_0) = y_0$, satisfies the bound

$$\|\, y(t) \,-\, \gamma_t(y_0)\, \| \;\leq\; \int_{\min(t_0,t)}^{\max(t_0,t)} \|\, b(s)\, \|\, ds. \tag{5.11}$$

Proof. For any $t \in \mathbb{R}$, let $x(t)$ be the solution of

$$\partial_t x(t) \;=\; A(t)\, x(t) \tag{5.12}$$

with boundary condition $x(0) = x_0$, and let γ_t be the linear mapping which takes x_0 to $x(t)$. By variation of constants, the solution of the inhomogeneous equation (5.10) may be expressed as

$$y(t) \;=\; \gamma_t \left(y_0 \,+\, \int_0^t (\gamma_s)^{-1}\, (b(s))\, ds \right). \tag{5.13}$$

The estimate (5.11) follows from (5.13) as $A(t)$ is norm preserving. □

Proof of Theorem 5.1: We prove (5.3) in two steps. First, we use Lemma 5.2 to establish a basic inequality, see (5.20) below. Next, using properties of the F-function, iteration of (5.20) yields (5.3) as claimed. Without loss of generality we may assume that $X, Y \subset \Lambda$.

First note that, the roles of A and B, and hence the roles their respective supports, X and Y, can be interchanged. This is due to the automorphism property of the dynamics, which gives

$$\|[\tau_{-t}^{\Lambda}(B), A]\| = \|\tau_t^{\Lambda}\left([\tau_{-t}^{\Lambda}(B), A]\right)\| = \|[\tau_t^{\Lambda}(A), B]\| \tag{5.14}$$

and the argument below can be applied to the left hand side of (5.14).

Therefore, without loss of generality, we can assume that

$$D(X,Y) = \sum_{x \in \partial_\Phi X} \sum_{y \in Y} F\left(d(x,y)\right) \tag{5.15}$$

To prove (5.20), consider the function

$$f(t) = \left[\tau_t^{\Lambda}\left(\tau_{-t}^{X}(A)\right), B\right], \tag{5.16}$$

where A and B are as in the statement of the theorem. Note that the inner dynamics, $\tau_{-t}^{X}(A)$, corresponds to evolution by the local Hamiltonian H_X, as defined e.g. in (4.3), with X being the support of the observable A. It is straightforward to verify

$$\begin{aligned} f'(t) &= i\left[\tau_t^\Lambda\left(\left[H_\Lambda - H_X, \tau_{-t}^X(A)\right]\right), B\right] \\ &= i \sum_{Z\in S_\Lambda(X)} \left[\left[\tau_t^\Lambda\left(\Phi(Z)\right), \tau_t^\Lambda\left(\tau_{-t}^X(A)\right)\right], B\right] \\ &= i \sum_{Z\in S_\Lambda(X)} \left[\tau_t^\Lambda\left(\Phi(Z)\right), f(t)\right] - i \sum_{Z\in S_\Lambda(X)} \left[\tau_t^\Lambda(\tau_{-t}^X(A)), \left[\tau_t^\Lambda\left(\Phi(Z)\right), B\right]\right]. \end{aligned} \tag{5.17}$$

where for the last equality we used the Jacobi identity. The first term in (5.17) above is norm preserving, and therefore, Lemma 5.2 implies that

$$\|[\tau_t^\Lambda(\tau_{-t}^X(A)), B]\| \leq \|[A,B]\| + 2\,\|A\| \sum_{Z\in S_\Lambda(X)} \int_{\min\{0,\mathrm{t}\}}^{\max\{0,\mathrm{t}\}} \left\|[\tau_s^\Lambda(\Phi(Z)), B]\right\| ds. \tag{5.18}$$

To ease notation, we will assume that $t \geq 0$ for the remainder of the argument. Changing the sign of t is equivalent to changing the sign of Φ and therefore leaves the estimate unchanged. For any $Z \in \mathcal{P}_0(\Gamma)$, introduce the quantity

$$C_B^\Lambda(Z;t) = \sup_{\substack{A\in\mathcal{A}_Z:\\ A\neq 0}} \frac{\|[\tau_t^\Lambda(A), B]\|}{\|A\|}. \tag{5.19}$$

Since $\mathrm{supp}(\tau_{-t}^X(A)) \subset X$ and $\|\tau_{-t}^X(A)\| = \|A\|$ (both for all $t \in \mathbb{R}$), the inequality (5.18) clearly implies

$$C_B^\Lambda(X;t) \leq C_B^\Lambda(X;0) + 2 \sum_{Z\in S_\Lambda(X)} \|\Phi(Z)\| \int_0^t C_B^\Lambda(Z;s)ds. \tag{5.20}$$

Note: it is clear from (5.20) that single-site interaction terms, i.e. those of the form $\Phi(\{z\})$ for some $z \in \Gamma$, do not contribute to this locality estimate.

The claim in (5.3) now follows from (5.20) by iteration. In fact, it is clear from the definition, see (5.19), that for any finite $Z \subset \Lambda$,

$$C_B^\Lambda(Z;0) \leq 2\,\|B\|\,\delta_Y(Z) \tag{5.21}$$

where δ_Y is defined by

$$\delta_Y(Z) = \begin{cases} 1 & \text{if } Z\cap Y \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{5.22}$$

To prove (5.3), let $N \geq 1$. Iteration of (5.20) using (5.21) yields

$$C_B^\Lambda(X;t) \leq 2\|B\| \left(\delta_Y(X) + \sum_{n=1}^N a_n \frac{(2t)^n}{n!}\right) + R_{N+1}(t) \tag{5.23}$$

where

$$a_n = \sum_{Z_1\in S_\Lambda(X)} \sum_{Z_2\in S_\Lambda(Z_1)} \cdots \sum_{Z_n\in S_\Lambda(Z_{n-1})} \left(\prod_{j=1}^n \|\Phi(Z_j)\|\right) \delta_Y(Z_n) \tag{5.24}$$

and

$$\begin{aligned} R_{N+1}(t) &= 2^{N+1} \sum_{Z_1\in S_\Lambda(X)} \sum_{Z_2\in S_\Lambda(Z_1)} \cdots \sum_{Z_{N+1}\in S_\Lambda(Z_N)} \int_0^t \int_0^{s_1} \cdots \int_0^{s_N} \times \\ &\quad \times \left(\prod_{j=1}^{N+1} \|\Phi(Z_j)\|\right) C_B^\Lambda(Z_{N+1}; s_{N+1}) ds_{N+1} ds_N \cdots ds_1 \end{aligned} \tag{5.25}$$

The remainder term $R_{N+1}(t)$ is estimated as follows. First, we bound $C_B^\Lambda(Z_{N+1}; s_{N+1})$ with $2\|B\|$ using its definition (5.19). Next, we note that the sums above are over chains of sets $(Z_1, Z_2, \cdots, Z_{N+1})$ which satisfy $Z_1 \cap \partial_\Phi X \neq \emptyset$ and $Z_j \cap Z_{j-1} \neq \emptyset$ for $2 \leq j \leq N+1$. Therefore, there are points $w_1, w_2, \cdots, w_{N+1} \in \Lambda$ with $w_1 \in Z_1 \cap \partial_\Phi X$ and $w_j \in Z_j \cap Z_{j-1}$ for all $2 \leq j \leq N+1$. A simple upper bound on these sums is then obtained by estimating

$$\sum_{Z_1 \in S_\Lambda(X)} \sum_{Z_2 \in S_\Lambda(Z_1)} \cdots \sum_{Z_{N+1} \in S_\Lambda(Z_N)} * \leq \sum_{w_1 \in \partial_\Phi X} \sum_{w_2, \ldots, w_{N+2} \in \Lambda} \sum_{\substack{Z_1, \ldots, Z_{N+1} \subset \Lambda: \\ w_k, w_{k+1} \in Z_k, k=1, \ldots, N+1}} * \tag{5.26}$$

where we have used that the last set Z_{N+1} must contain more than one point since $Z_{N+1} \in S_\Lambda(Z_N)$. By (4.44)

$$\sum_{\substack{Z_k \subset \Lambda: \\ w_k, w_{k+1} \in Z_k}} \|\Phi(Z_k)\| \leq \|\Phi\|_F F(d(w_k, w_{k+1})) \tag{5.27}$$

and the convolution property, we find that

$$\begin{aligned} R_{N+1}(t) &\leq 2\|B\| \cdot \frac{(2t)^{N+1}}{(N+1)!} \sum_{Z_1 \in S_\Lambda(X)} \sum_{Z_2 \in S_\Lambda(Z_1)} \cdots \sum_{Z_{N+1} \in S_\Lambda(Z_N)} \left(\prod_{j=1}^{N+1} \|\Phi(Z_j)\| \right) \\ &\leq 2\|B\| \cdot \frac{(2t\|\Phi\|_F)^{N+1}}{(N+1)!} \sum_{w_1 \in \partial_\Phi X} \sum_{w_2, \ldots, w_{N+2} \in \Lambda} \prod_{k=1}^{N+1} F(d(w_k, w_{k+1})) \\ &\leq \frac{2\|B\|}{C_F} \cdot \frac{(2t\|\Phi\|_F C_F)^{N+1}}{(N+1)!} \sum_{w_1 \in \partial_\Phi X} \sum_{w_{N+2} \in \Lambda} F(d(w_1, w_{N+2})) \end{aligned} \tag{5.28}$$

Since F is uniformly summable and X is finite, this bound clearly shows that $R_{N+1}(t)$ goes to 0 as $N \to \infty$. We have proven that

$$C_B^\Lambda(X; t) \leq 2\|B\| \sum_{n=1}^{\infty} a_n \cdot \frac{(2t)^n}{n!} \tag{5.29}$$

The coefficients a_n are bounded similarly. In fact, using the additional constraint that the final set Z_n must intersect Y, we find that

$$a_n \leq \frac{(C\|\Phi\|_F)^n}{C_F} \sum_{y \in Y} \sum_{x \in \partial_\Phi X} F(d(x, y)) \tag{5.30}$$

and therefore,

$$C_B(X; t) \leq \frac{2\,\|B\|}{C_F} \left(e^{2C_F\|\Phi\|_F t} - 1 \right) \sum_{y \in Y} \sum_{x \in \partial_\Phi X} F(d(x, y)), \tag{5.31}$$

□

In combination with Lemma 4.1, the Lieb-Robinson bounds of Theorem 5.1 show that the time evolution of a local observable with support in $X \in \mathcal{P}_0(\Gamma)$, yields an observable which, up to a small correction, is localized in a larger but still finite region. More precisely, for $X \in \mathcal{P}_0(\Gamma)$, and with $\|\Phi\|_a < \infty$, for some $a > 0$ and $v_{\Phi,a}$ is the quantity defined in (5.7), define

$$X(v_{\Phi,a}|t| + r) = \{x \in \Gamma \mid d(x, X) \leq v_{\Phi,a}|t| + r\}.$$

Then, for $A \in \mathcal{A}_X$, define

$$(\tau_t^\Lambda(A))_{X(v_{\Phi,a}|t|+r)} = \mathrm{Tr}_{\mathcal{H}_{\Lambda \setminus X(v_{\Phi,a}|t|+r)}} \tau_t^\Lambda(A).$$

Then $(\tau_t^\Lambda(A))_{X(v_{\Phi,a}|t|+r)} \in \mathcal{A}_{X(v_{\Phi,a}|t|+r)} \subset \mathcal{A}_\Lambda$ and

$$\|\tau_t^\Lambda(A) - (\tau_t^\Lambda(A))_{X(v_{\Phi,a}|t|+r)}\| \leq \frac{2\,\|A\|\,|X|}{C_a}\,\|F\|\,e^{-ar}.$$

This observation, and the fact that the bound above is uniform in Λ, is a clear indication that the dynamics of local observables under the interaction with the infinite system on Γ, should be well-defined. We prove that this is indeed the case in the next section.

5.1. **Existence of the Dynamics.** Lieb-Robinson bounds can be used to establish the existence of a limiting dynamics for interactions $\Phi \in \mathcal{B}_F(\Gamma)$. To see this we will consider limits of the finite volume dynamics along increasing, exhaustive sequences $\{\Lambda_n\}$, i.e., for all $n \geq 1$, $\Lambda_n \in \mathcal{P}_0(\Gamma)$, $\Lambda_n \subset \Lambda_{n+1}$, and for any $z \in \Gamma$, there exists an $n \geq 1$ for which $z \in \Lambda_n$.

Theorem 5.3. *Let $\Phi \in \mathcal{B}_F(\Gamma)$. Along any increasing, exhaustive sequence $\{\Lambda_n\}$ of finite subsets of Γ, the norm limit*

$$\tau_t(A) = \lim_{n\to\infty} \tau_t^{\Lambda_n}(A) \tag{5.32}$$

exists for all $t \in \mathbb{R}$ and $A \in \mathcal{A}_\Gamma^{\rm loc}$. The convergence in (5.32) is uniform for t in compact sets, and moreover, it is independent of the choice of exhaustive sequence $\{\Lambda_n\}$. The collection $\{\tau_t\}_{t\in\mathbb{R}}$, which we denote by the infinite volume dynamics corresponding to Φ, defines a strongly continuous, one parameter group of $*$-automorphisms on $\mathcal{A}_\Gamma$.

Proof. Let $A \in \mathcal{A}_\Gamma^{\rm loc}$ and denote by $X = \text{supp}(A) \in \mathcal{P}_0(\Gamma)$. Take $m \geq 1$ large enough so that $X \subset \Lambda_m$. For any $n \geq m$, we have that,

$$\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \;=\; \int_0^t \frac{d}{ds}\left(\tau_s^{\Lambda_n}\left(\tau_{t-s}^{\Lambda_m}(A)\right)\right)\,ds, \tag{5.33}$$

and since

$$\frac{d}{ds}\left(\tau_s^{\Lambda_n}\left(\tau_{t-s}^{\Lambda_m}(A)\right)\right) = i\tau_s^{\Lambda_n}\left(\left[H_{\Lambda_n} - H_{\Lambda_m}, \tau_{t-s}^{\Lambda_m}(A)\right]\right)\,, \tag{5.34}$$

it is clear that for $t > 0$

$$\|\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)\| \;\leq\; \sum_{Z\in S_{\Lambda_n}(\Lambda_m)} \int_0^t \|[\tau_s^{\Lambda_m}(A), \Phi(Z)]\|\,ds. \tag{5.35}$$

(Again, analogous results hold for $t \leq 0$.) The estimate continues by dividing the above sum on Z:

$$\begin{aligned}
\|\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)\| \;&\leq\; 2\|A\|t \sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X\neq\emptyset}} \|\Phi(Z)]\| + \sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X=\emptyset}} \int_0^t \|[\tau_s^{\Lambda_m}(A), \Phi(Z)]\|\,ds \\
&\leq\; 2\|A\|t \sum_{x\in X} \sum_{z\in\Lambda_n\setminus\Lambda_m} \sum_{\substack{Z\subset\Lambda_n:\\ x,z\in Z}} \|\Phi(Z)\| \\
&\quad + \frac{2\|A\|}{C} \int_0^t (e^{2\|\Phi\|_F C_F s} - 1)\,ds \sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X=\emptyset}} \|\Phi(Z)\| \sum_{x\in X, z\in Z} F(d(x,z))
\end{aligned} \tag{5.36}$$

where we have used Theorem 5.1 on the second sum above. Observe that

$$\begin{aligned}
\sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X=\emptyset}} \|\Phi(Z)\| \sum_{x\in X, z\in Z} F(d(x,z)) &\leq \sum_{x\in X}\sum_{z'\in\Lambda_n\setminus\Lambda_m}\sum_{z\in\Lambda_n} F(d(x,z)) \sum_{\substack{Z\subset\Lambda_n:\\ z,z'\in Z}} \|\Phi(Z)\| \\
&\leq \|\Phi\|_F \sum_{x\in X}\sum_{z'\in\Lambda_n\setminus\Lambda_m}\sum_{z\in\Lambda_n} F(d(x,z))F(d(z,z')) \\
&\leq C\|\Phi\|_F \sum_{x\in X}\sum_{z'\in\Lambda_n\setminus\Lambda_m} F(d(x,z')) \qquad (5.37)
\end{aligned}$$

Altogether then, we have shown that

$$\|\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A)\| \leq 2\|A\|\|\Phi\|_F \left(\int_0^t e^{2\|\Phi\|_F C_F s} ds\right) \sum_{x\in X}\sum_{z\in\Lambda_n\setminus\Lambda_m} F(d(x,z)) \qquad (5.38)$$

This proves that the sequence of finite volume evolutions is Cauchy and hence convergent; at least on $\mathcal{A}_\Gamma^{\mathrm{loc}}$. The remaining claims follow by elementary arguments. In particular, see the Exercise 15 of Sections 1-3. □

By general arguments of semi-group theory (see, e.g., [3][Proposition 6.2.3]), the strongly continuous, one-parameter group of $*$-automorphisms $\{\tau_t\}_{t\in\mathbb{R}}$ is generated by a closed operator δ in the following sense. For all $A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$, using an estimate in terms of $\|\Phi\|_F$, one can show the existence of the limit in $\mathcal{A}_\Gamma$:

$$\delta(A) = \lim_{\Lambda\to\Gamma} [H_\Lambda, A]. \qquad (5.39)$$

and δ is the closure of this operator, meaning the domain of the generally unbounded operator δ, $\mathrm{dom}(\delta)$, contains $\mathcal{A}_\Gamma^{\mathrm{loc}}$ as a core. For $A \in \mathrm{dom}(\delta)$, we have

$$\frac{d}{dt}\tau_t(A) = i\delta(\tau_t(A)) = i\tau_t(\delta(A)), \qquad (5.40)$$

and it is customary to write $\tau_t = e^{it\delta}$.

6. Ground States and Equilibrium States

For a quantum spin system associated with a finite set Λ, the Hamiltonian $H_\Lambda = H_\Lambda^* \in \mathcal{A}_\Lambda$ is a self-adjoint operator acting on a finite-dimensional complex Hilbert space $\mathcal{H}_\Lambda$. Its spectrum is a finite set of real eigenvalues and a *ground state* is defined to be any state ω_0 that minimizes the energy, *i.e.* , such that

$$\omega_0(H_\Lambda) = \min\{\omega(H_\Lambda) \mid \omega \text{ a state on } \mathcal{A}_\Lambda\}. \tag{6.1}$$

It is a simple exercise to show that ground states of finite quantum spin systems are exactly those states that have density matrices with a range that is a subspace of the eigenspace corresponding to the smallest eigenvalue of H_Λ. The ground state of the system is unique if and only if this eigenvalue is simple.

For a finite quantum spin system with Hamiltonian H_Λ, thermal equilibrium at inverse temperature $\beta \in [0, \infty)$ is the unique state, the *Gibbs state*, given by the density matrix ρ_β defined as follows:

$$\rho_\beta = \frac{1}{Z_\beta} e^{-\beta H_\Lambda}, \quad Z_\beta = \mathrm{Tr} e^{-\beta H_\Lambda}. \tag{6.2}$$

In the sections below we derive some basic properties and equivalent characterizations of ground states and equilibrium states, which will be useful to formulate these concepts in the infinite system setting.

6.1. **Ground States.** Let $E_0(\Lambda)$ denote the smallest eigenvalue of the Hamiltonian H_Λ on a finite-dimensional Hilbert space $\mathcal{H}_\Lambda$. In this context $E_0(\Lambda)$ is the *ground-state energy*, and any state ω on $\mathcal{A}_\Lambda$ such that $\omega(H_\Lambda) = E_0(\Lambda)$ is a *ground state*. In the following proposition it is shown that the set of all ground states for the system with Hamiltonian H_Λ is the set of all states satisfying $\omega(A^*[H_\Lambda, A]) \geq 0$, for all $A \in \mathcal{A}_\Lambda$, and that this property is inherited by limits of sequences of ground states of finite-volume systems defined on a sequence $\Lambda_n \in \mathcal{P}_0(\Gamma)$, defined by an interaction $\Phi \in \mathcal{B}_F(\Gamma)$, for any F-function F.

It is clear that to any sequence of finite volumes $\{\Lambda_n\}$ with $\Lambda_n \to \Gamma$, there correspond one or more sequences of ground-states ω_{Λ_n}. Using the Banach-Alaoglu theorem, it is easy to see that each of these sequences has an accumulation point (and a convergent subsequence).

Note that there is no simple definition of a limit of the Hilbert spaces $\mathcal{H}_{\Lambda_n}$, nor of the Hamiltonians H_{Λ_n}, but it makes perfect sense to consider the limiting states on $\mathcal{A}_\Gamma^{\mathrm{loc}}$, and, by unique continuous extension, also on $\mathcal{A}_\Gamma$. This provides a good option for defining *ground states* for the infinite volume Γ.

Proposition 6.1.
(i) Let ω be a state of a system with Hamiltonian H on a finite-dimensional Hilbert space $\mathcal{H}$, i.e. , suppose that the range of the density matrix ρ of ω is a subspace of the eigenspace of H corresponding to its smallest eigenvalue E_0. Then we have

$$\omega(A^*[H, A]) \geq 0, \ \text{for all } A \in \mathcal{B}(\mathcal{H}). \tag{6.3}$$

Conversely, every state on $\mathcal{B}(\mathcal{H})$ satisfying (6.3) is a ground state of the system.
(ii) Let $\{\Lambda_n\}$ be an exhaustive sequence of finite volumes in Γ. For each $n \geq 1$, let ω_n be a ground state of H_{Λ_n}. If

$$\omega_n(A) \to \omega(A) \quad \text{for all } A \in \mathcal{A}_\Gamma^{\mathrm{loc}} \tag{6.4}$$

and

$$\delta_{\Lambda_n}(A) := [H_{\Lambda_n}, A] \to \delta(A) \quad \text{for all } A \in \mathcal{A}_\Gamma^{\mathrm{loc}} \tag{6.5}$$

in the strong sense (i.e. in norm), then ω is a state on $\mathcal{A}_\Gamma^{\mathrm{loc}}$ satisfying

$$\omega(A^*\delta(A)) \geq 0, \ \text{for all } A \in \mathcal{A}_\Gamma^{\mathrm{loc}}. \tag{6.6}$$

Proof. (i). Observe that

$$\begin{aligned} \omega(A^*[H,A]) &= \mathrm{Tr}\rho A^*[H,A] \\ &= \mathrm{Tr}\rho A^*HA - \mathrm{Tr}H\rho A^*A = \mathrm{Tr}\rho A^*(H-E_0\mathbb{1})A \geq 0 \end{aligned} \tag{6.7}$$

where we have used $H\rho = E_0\rho$ and $H_\Lambda \geq E_0\mathbb{1}$.

To prove the converse, assume that there exists $0 \neq \psi \in \mathrm{ran}\rho$ such that $H\psi \neq E_0\psi$ and consider $A = |\psi_0\rangle\langle\psi|$ where $\psi_0 \neq 0$ and $H\psi_0 = E_0\psi_0$. An easy calculation shows that $\omega(A^*[H,A]) < 0$. Therefore, ω does not satisfy (6.3).

(ii). To study the limit $n \to \infty$, we write

$$\begin{aligned} &\omega(A^*\delta(A)) - \omega_{\Lambda_n}(A^*\delta_{\Lambda_n}(A)) \\ &= \omega\left(A^*\left(\delta(A) - \delta_{\Lambda_n}(A)\right)\right) + (\omega - \omega_{\Lambda_n})\left(A^*\left(\delta_{\Lambda_n}(A) - \delta(A)\right)\right) + (\omega - \omega_{\Lambda_n})(A^*\delta(A)). \end{aligned} \tag{6.8}$$

Using the boundedness of states as linear functionals and the assumptions, we see that all terms on the last line vanish in the limit $n \to \infty$, and the claimed result follows. □

Note that there are in general many sequences H_{Λ_n} such that (6.5) holds with the same generator δ. The limits of ground states of each such sequence of finite-volume Hamiltonians will all satisfy (6.3) and it will make sense to consider them all as ground states of the C^*-dynamical system $(\mathcal{A}_\Gamma, \{\tau_t = e^{it\delta} \mid t \in \mathbb{R}\})$. Therefore, we adopt the following definition.

Definition 6.2. Let $\mathcal{A}$ be a C^*-algebra and $\{\tau_t = e^{it\delta} \mid t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of automorphisms of $\mathcal{A}$. Then, a state ω on $\mathcal{A}$ is a ground-state for τ_t if

$$\omega(A^*\delta(A)) \geq 0 \quad \text{for all } A \in \mathrm{dom}(\delta). \tag{6.9}$$

Note that it suffices to require (6.9) for A in a core for δ. In the context of quantum spin systems defined by an interaction $\Phi \in \mathcal{B}_F(\Gamma)$, a convenient core for δ is given by the local observables $\mathcal{A}_\Gamma^{\mathrm{loc}}$.

The next proposition shows that ground states are time-invariant, *i.e.* , $\omega \circ \tau_t = \omega$, for all $t \in \mathbb{R}$.

Proposition 6.3. *Let $\mathcal{A}$ be a C^*-algebra and $\{\tau_t = e^{it\delta} \mid t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of automorphisms of $\mathcal{A}$. Then, the following three conditions are equivalent:*

i. $\omega \circ \tau_t = \omega$, *for all* $t \in \mathbb{R}$;
ii. $\omega(\delta(A)) = 0$, *for all* $A \in \mathrm{dom}(\delta)$;
iii. $\omega(A^*\delta(A)) \in \mathbb{R}$, *for all* $A \in \mathrm{dom}(\delta)$.

We leave the proof of this proposition as an exercise for the reader. (Hint: $\omega(\delta(A^*A)) = 2i\mathrm{Im}[\omega(A^*\delta(A))]$.)

6.2. **Thermal Equilibrium, the Free Energy, and the Variational Principle for Gibbs States.** For a finite spin systems, the thermal equilibrium state at inverse temperature $\beta \in [0,\infty)$, can be defined as the minimizer of the *free energy* functional. This provides a definition of equilibrium states analogous to the definition of ground states as those states that minimize the energy. As we shall see, ground states correspond to zero temperature, *i.e.* , $\beta = +\infty$.

In order to define the free energy functional, we start with von Neumann's definition of the *entropy*, $S(\rho)$, of a state defined by a density matrix ρ:

$$S(\rho) = -\mathrm{Tr}\rho\log\rho. \tag{6.10}$$

Here $\rho\log\rho$ is defined through the functional calculus with the continuous function $x\log x : [0,1] \to \mathbb{R}$. When $\mathcal{H}$ is finite-dimensional the entropy is finite for all ρ and satisfies the bound

$$0 \leq S(\rho) \leq \log\dim\mathcal{H}, \tag{6.11}$$

the proof of which we leave as an exercise.

Let $H = H^* \in \mathcal{B}(\mathcal{H})$ be the Hamiltonian of a finite quantum spin system. The *Gibbs state* at inverse temperature $\beta \in (0, \infty)$ for the system with Hamiltonian H is defined by the density matrix

$$\rho_\beta = \frac{1}{Z(\beta)} e^{-\beta H}, \text{ with } Z(\beta) = \mathrm{Tr} e^{-\beta H}. \tag{6.12}$$

The normalization factor $Z(\beta)$ is called the *partition function.* We will denote the Gibbs state by ω_β. The parameter β corresponds to the temperature T in the sense that $\beta = (k_B T)^{-1}$, where k_B is Boltzmann's constant. Thus $T = 0$ corresponds to $\beta \to \infty$, and in turn to the ground state.

For $\beta \in (0, \infty)$, the free-energy functional F_β is given by

$$F_\beta(\rho) = \mathrm{Tr} \rho H - \beta^{-1} S(\rho). \tag{6.13}$$

Proposition 6.4. *ρ_β is the unique density matrix that minimizes F_β, i.e. , for all density matrices ρ we have*

$$F_\beta(\rho_\beta) \leq F_\beta(\rho), \text{ and } F_\beta(\rho) = F_\beta(\rho_\beta) \Rightarrow \rho = \rho_\beta. \tag{6.14}$$

Using this proposition, the minimum value of F_β is easily seen to be given by $f(\beta) = -\beta^{-1} \log Z(\beta)$ and is called the *free energy* of the system at inverse temperature β.

The proof of this proposition follows from a simple application of the following lemma.

Lemma 6.5 (Klein [17], Ohya-Petz [15]). *Let A and B be two non-negative definite matrices satisfying $0 \leq A, B \leq \mathbb{1}$ and such that $\ker B \subset \ker A$. Then*

$$\mathrm{Tr} A(\log A - \log B) \geq \mathrm{Tr}(A - B) + \frac{1}{2} \mathrm{Tr}(A - B)^2 \tag{6.15}$$

Proof. The function $f(x) = -x \log x$, $x > 0$, continuously extended such that $f(0) = 0$, is easily seen to be concave. In fact $f \in C^2((0, \infty))$ with

$$f''(x) = -\frac{1}{x} \tag{6.16}$$

By the Taylor Remainder Theorem and the expression for f'', it follows that for all x and y such that $0 \leq x < y \leq 1$, there exists a ξ such that $x \leq \xi \leq y$ and

$$f(y) - f(x) - (y - x) f'(y) = -\frac{1}{2}(x - y)^2 f''(\xi) \geq \frac{1}{2}(x - y)^2 \tag{6.17}$$

As A and B are non-negative definite, they are diagonalizable. Denote their eigenvalues by a_i and b_i, and the corresponding orthonormal eigenvectors by φ_i and ψ_i, respectively. From the assumptions it follows that $0 \leq a_i, b_i \leq 1$. Using the spectral decompositions of A and B, i.e.,

$$A = \sum_i a_i |\varphi_i\rangle\langle\varphi_i| \tag{6.18}$$

$$B = \sum_i b_i |\psi_i\rangle\langle\psi_i| \tag{6.19}$$

$$\sum_i |\varphi_i\rangle\langle\varphi_i| = \sum_i |\psi_i\rangle\langle\psi_i| = \mathbb{1} \tag{6.20}$$

we see that

$$\begin{aligned}
&\mathrm{Tr}A(\log A - \log B) - \mathrm{Tr}(A-B) - \frac{1}{2}\mathrm{Tr}(A-B)^2 \\
&= \sum_{ij} \mathrm{Tr}|\psi_i\rangle\langle\psi_i||\varphi_j\rangle\langle\varphi_j| \left[-f(A) + f(B) + (A-B)f'(B) - \frac{1}{2}(A^2+B^2-2AB)\right] \\
&= \sum_{ij} \mathrm{Tr}|\psi_i\rangle\langle\psi_i||\varphi_j\rangle\langle\varphi_j| \left[-f(a_j) + f(b_i) + (a_j-b_i)f'(b_i) - \frac{1}{2}(a_j-b_i)^2\right] \\
&\geq 0
\end{aligned}$$

where the last inequality follows from applying (6.17) term by term. □

Now to prove Proposition 6.4, we can apply Lemma 6.5 with $A = \rho$, where ρ is an arbitrary density matrix, and $B = \rho_\beta$. Note than $\ker B = \{0\}$. This gives

$$\beta(f_\beta - F(\beta)) = \mathrm{Tr}\rho\log\rho - \mathrm{Tr}\rho\log\left(\frac{e^{-\beta H}}{Z(\beta)}\right) \tag{6.21}$$

$$\geq \frac{1}{2}\mathrm{Tr}(\rho-\rho_\beta)^2 \geq 0 \tag{6.22}$$

If the RHS vanishes, we have $\rho = \rho_\beta$. Hence the minimum of F_β is uniquely attained for $\rho = \rho_\beta$.

6.3. **The Kubo-Martin-Schwinger condition.** Again, we consider a finite-dimensional Hilbert space $\mathcal{H}$, and a Hamiltonian $H = H^* \in \mathcal{A} = \mathcal{B}(\mathcal{H})$. Denote the Heisenberg dynamics by τ_t. Since H is bounded it is straightforward to define the analytic continuation of $\tau_t(A)$ for all $t \in \mathbb{C}$. A state ω on $\mathcal{A}$ is called a β-KMS state if, for all $A, B \in \mathcal{A}$, it satisfies

$$\omega(A\tau_{i\beta}(B)) = \omega(BA). \tag{6.23}$$

Proposition 6.6. *ω is a β-KMS state if and only if $\omega = \omega_\beta$, i.e. , the KMS state coincides with the Gibbs state.*

Proof. First, the KMS property of the Gibbs state follows from a simple computation using the dynamics and the cyclicity of the trace:

$$\mathrm{Tr}e^{-\beta H}Ae^{itH}Be^{-itH}|_{t=i\beta} = \mathrm{Tr}e^{-\beta H}Ae^{-\beta H}Be^{\beta H} = \mathrm{Tr}e^{-\beta H}BA. \tag{6.24}$$

For the other direction, we use an orthonormal basis of eigenvectors of H, $|1\rangle, \ldots, |n\rangle$, with eigenvalues $\lambda_1, \ldots, \lambda_n$. The β-KMS property of state with density matrix ρ with $A = |i\rangle\langle i|j$ and $B = |k\rangle\langle k|l$ then reads

$$\mathrm{Tr}\rho|i\rangle\langle i|je^{\beta(\lambda_l-\lambda_k}|k\rangle\langle k|l = \mathrm{Tr}\rho|k\rangle\langle k|l|i\rangle\langle i|j, \tag{6.25}$$

which translates into

$$\delta_{k,j}\langle l \mid \rho \mid i\rangle e^{\beta(\lambda_l-\lambda_k)} = \delta_{i,l}\langle j \mid \rho \mid k\rangle. \tag{6.26}$$

since ρ is a density matrix, at least on of its diagonal matrix elements is non-zero, say $\langle i \mid \rho \mid i\rangle > 0$. With $l = i$ and $k \neq j$, the above relation implies that ρ is in fact diagonal in the eigenbasis of H. Finally, with $l = i$ and $k = j$, we obtain that there is a constant c such that

$$\langle i \mid \rho \mid i\rangle e^{\beta\lambda_i} = c. \tag{6.27}$$

It follows that ρ is the Gibbs state. □

Note that the arguments in the proof of this proposition rely on the finite-dimensionality of the Hilbert space.

6.4. **The Energy-Entropy Balance inequalities.** A third criterion for thermal equilibrium is expressed by a family of inequalities called the Energy-Entropy- Balance (EEB) inequalities. Again, for now we only consider finite systems and define $\delta(\cdot) = [H, \cdot]$. Then, we say that ω satisfies EEB at inverse temperature β if, for all $X \in \mathcal{A}$

$$\omega\left(X^*[H, X]\right) \geq \beta^{-1}\omega(X^*X)\log\left(\frac{\omega(X^*X)}{\omega(XX^*)}\right) \quad \text{for all } X \in \mathcal{A}. \tag{6.28}$$

Our next goal is to prove that, for a finite system, the EEB inequalities uniquely characterize the Gibbs state. First we prepare some auxiliary material that is also useful more generally.

The formulation of the EEB inequalities uses the function $f : [0, +\infty) \times [0, +\infty) \to (-\infty, +\infty]$ defined by

$$f(x, y) = \begin{cases} x\log\frac{x}{y} & \text{if } x, y > 0 \\ 0 & \text{if } x = 0, y \geq 0 \\ +\infty & \text{if } x > 0, y = 0 \end{cases} \tag{6.29}$$

In the following, whenever we write something of the form $x\log(x/y)$, we mean f as defined above. We will use the following elementary properties of f.

Proposition 6.7. *The function f defined in (6.29) has the following properties:*
(i) f is lower semicontinuous.
(ii) f is jointly convex in (x, y).
(iii) f is homogeneous of degree one. i.e., for all $\lambda \geq 0$,

$$f(\lambda x, \lambda y) = \lambda f(x, y)$$

(iv) For all finite sequences of non-negative numbers t_i, x_i, y_i, $i = 1, \ldots, n$, one has

$$f(\sum_i t_i x_i, \sum_i t_i y_i) \leq \sum_i t_i f(x_i, y_i).$$

We leave the proof of this proposition as an exercise for the reader.

Theorem 6.8 ([5,6]). *Let ω be a state on $\mathcal{A}$. The following are equivalent conditions:*
(i) ω is the Gibbs state corresponding to H and inverse temperature β.
(ii) For all $X \in \mathcal{A}$ one has

$$\beta\omega(X^*[H, X]) \geq \omega(X^*X)\log\frac{\omega(X^*X)}{\omega(XX^*)} = f(\omega(X^*X), \omega(XX^*)) \tag{6.30}$$

It is worth noting that the equilibrium condition expressed by the EEB is equivalent to the KMS condition in the general context of C^*-dynamical systems (see [3][Theorem 5.3.15]).

In the context of finite quantum spin systems, the theorem says that the Gibbs state satisfies the inequalities (6.30) for all $X \in \mathcal{A}$, and that it is the only state that does so. We will derive this property from the variational principle following a common procedure: we will define suitable curves in the space of all states that pass through the Gibbs state, or emanate from it, and compute and estimate the derivative of the free energy functional along these curves. The EEB inequalities will follow from expressing that the state ω minimizes the free energy functional. The converse direction will be proved by explicit computation. In order to define curves in the space of all states we use a class of semigroups on $\mathcal{A}$ described in the next section. Their definition and essential properties are as follows.

Let $X \in \mathcal{B}(\mathcal{H})$. Define $L_X : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, by

$$L_X(A) = X^*AX - \frac{1}{2}(X^*XA + AX^*X)$$

Clearly, as $\|L_X(A)\| \leq 2\|X\|^2\|A\|$, L_X is a bounded linear transformation on the Banach space $\mathcal{B}(H)$. Therefore, we can define

$$\gamma_t(A) = e^{tL_X}(A)$$

$(\gamma_t)_{t\geq 0}$ is a semigroup with the following properties: $\gamma_t(\mathbb{1}) = \mathbb{1}$, and $\gamma_t(A)$ is positive for all $t \geq 0$ and $A \geq 0$. For a prove of this and other important properties see, e.g., the lecture notes by Michael Wolf [20]. A map γ_t with this property is called a positive map and L_X generates a semigroup of such maps. From these properties it immediately follows that, for all t, there is a unique density matrix ρ_t such that

$$\mathrm{Tr}\rho_t A = \mathrm{Tr}\rho\gamma_t(A)$$

ρ_t is obtained from ρ by application of another semigroup of positive maps, γ_t^*, which are the adjoints of γ_t with respect to the Hilbert-Schmidt inner product on $\mathcal{B}(\mathcal{H})$. Its generator is given by

$$L_X^*(\rho) = X\rho X^* - \frac{1}{2}(X^*X\rho + \rho X^*X).$$

In the finite-dimensional context, γ_t is a well-defined bounded linear transformation on $\mathcal{A}$ for all $t \in \mathbb{R}$. The norm of it, however, diverges as $t \to -\infty$. So although we have curves ρ_t, in the space of density matrices defined for all $t \in \mathbb{R}$, we will only use $t \geq 0$. In infinite-dimensional situations γ_t is in general not defined for $t < 0$.

Proof of Theorem 6.8. The proof of the EEB inequalities consists in deriving the following two relations:

$$\lim_{t\downarrow 0} \frac{\mathrm{Tr}\rho_t H - \mathrm{Tr}\rho_\beta H}{t} = \omega_\beta(X^*[H,X]) \tag{6.31}$$

$$\lim_{t\downarrow 0} \frac{S(\rho_t) - S(\rho_\beta)}{t} \geq \omega_\beta(X^*X)\log\frac{\omega_\beta(X^*X)}{\omega_\beta(XX^*)} \tag{6.32}$$

Here, $\rho_\beta = \rho_0$, and $\omega_\beta(A) = \mathrm{Tr}\rho_\beta A$. The EEB inequalities then follow from the Variational Principle. Since

$$F_\beta(\rho_t) - F_\beta(\rho_\beta) \geq 0$$

and therefore, for all $t > 0$, we must have

$$\frac{\mathrm{Tr}\rho_t H - \mathrm{Tr}\rho_\beta H}{t} \geq \frac{1}{\beta}\frac{S(\rho_t) - S(\rho_\beta)}{t}$$

Below we take the limit $t \downarrow 0$, compute the LHS and prove a lower bound for the RHS. The resulting inequalities will turn out to be the EEB inequalities.

The derivative of the energy is easy to compute:

$$\frac{d}{dt}\omega(\gamma_t(H))\big|_{t=0} = \omega(L_X(H)) = \mathrm{Tr}\rho_\beta X^*HX - \frac{1}{2}\mathrm{Tr}\rho_\beta(X^*XH + HX^*X)$$

We are interested in the derivative in $\rho = \rho_\beta$. As $[\rho_\beta, H] = 0$, the last two terms are equal and can be combined. The result is (6.31).

For the entropy term we will need to differentiate operator valued functions of the type $\log A_t$. This is non-trivial. Usually the log function is defined by its series expansion around $\mathbb{1}$. To compute the derivative we will use the identity

$$\log x = \int_0^\infty \left[\frac{1}{1+t} - \frac{1}{x+t}\right] dt$$

for $x > 0$. So, for invertible $A_t \geq 0$, we consider

$$\begin{aligned}\frac{d}{dt}\log A_t &= \frac{d}{dt}\int_0^\infty \left[\frac{1}{1+s} - \frac{1}{A_t+s}\right] ds \\ &= \int_0^\infty (A_t+s)^{-1}\left(\frac{d}{dt}A_t\right)(A_t+s)^{-1} ds\end{aligned}$$

Here, we used the operator identity $A^{-1}(B-A)B^{-1} = A^{-1} - B^{-1}$ to compute

$$\frac{d}{dt}(A_t)^{-1} = -A_t^{-1}\left(\frac{d}{dt}A_t\right)A_t^{-1}$$

When we apply this to $-S(\rho_t)$ we get

$$\begin{aligned}\mathrm{Tr}\rho\frac{d}{dt}\log\rho_t\big|_{t=0} &= \mathrm{Tr}\rho\int_0^\infty \frac{1}{\rho+t}L_X^*(\rho)\frac{1}{\rho+t}dt \\ &= \mathrm{Tr}\rho\rho^{-1}L_X^*(\rho) \\ &= \mathrm{Tr}L_X^*(\rho)\end{aligned}$$

Now we can compute the derivative of the entropy term:

$$\begin{aligned}\frac{d}{dt}S(\rho_t)\big|_{t=0} &= -\mathrm{Tr}\frac{d}{dt}\rho_t\big|_{t=0} - \mathrm{Tr}\rho_t\frac{d}{dt}\log(\rho_t)\big|_{t=0} \\ &= -\mathrm{Tr}L_X^*(\rho)\log\rho - \mathrm{Tr}L_X^*(\rho) \\ &= -\mathrm{Tr}L_X^*(\rho)\log\rho\end{aligned}$$

where we used that $\mathrm{Tr}L_X^*(\rho) = \mathrm{Tr}\rho L_X(\mathbb{1}) = 0$.

Now we have to estimate (6.33). We will prove that

$$\begin{aligned}-\mathrm{Tr}\rho L_X(\log\rho) &= -\mathrm{Tr}\rho X^*(\log\rho)X + \frac{1}{2}\mathrm{Tr}\rho X^*X\log\rho + \frac{1}{2}\mathrm{Tr}\rho(\log\rho)X^*X \\ &\geq f(\mathrm{Tr}\rho X^*X, \mathrm{Tr}\rho XX^*)\end{aligned}$$

where f is the function defined in (6.29). To this end we use the spectral decomposition of ρ:

$$\rho = \sum_i \rho_i|\phi_i\rangle\langle\phi_i|\phi_i$$

Using this we can write the LHS of the inequality as follows:

$$-\sum_{ij}\rho_i\langle\phi_i, X^*\phi_j\rangle\log\rho_j\langle\phi_j, X\phi_i\rangle + \sum_{ij}\rho_i\log\rho_i\langle\phi_i, X^*\phi_j\rangle\langle\phi_j, X\phi_i\rangle$$

If we let a_{ij} denote the matrix elements $\langle\phi_j, X\phi_i\rangle$, this can be written as

$$\sum_{ij} f(\rho_i,\rho_j)|a_{ij}|^2$$

Property (iv) of Proposition 6.7 then yields

$$\begin{aligned}-\mathrm{Tr}\rho L_X(\log\rho) &\geq f(\sum_{ij}\rho_i|a_{ij}|^2, \sum_{ij}\rho_j|a_{ij}|^2) \\ &= f(\mathrm{Tr}\rho X^*X, \mathrm{Tr}\rho XX^*)\end{aligned}$$

This concludes the proof of (i) $\Rightarrow$ (ii) in Theorem 6.8.

The opposite direction proceeds by solving the EEB inequalities. Suppose the Hamiltonian has eigenvalues λ_i and an orthonormal basis of eigenvectors ϕ_i. We will use the basis E_{ij} for the matrices:

$$E_{ij} = |\phi_i\rangle\langle\phi_i|\phi_j, \quad E_{ij}^* = E_{ji}, \quad E_{ij}E_{kl} = \delta_{jk}E_{il} \quad .$$

The spectral decomposition of the Hamiltonian can then be written as

$$H = \sum_i \lambda_i E_{ii}.$$

First, we note that if ω satisfies (6.30), then the corresponding density matrix commutes with the Hamiltonian. This follows from the fact that the inequalities imply that, for all X,

$$\mathrm{Tr}\rho X^* H X - \mathrm{Tr}\rho X^* X H \in \mathbb{R}$$

and, as

$$\mathrm{Im}\ \mathrm{Tr}\rho X^* H X - \mathrm{Tr}\rho X^* X H = \mathrm{Tr} X^* X[\rho, H]$$

for arbitrary $X \in \mathcal{A}$, this implies $[\rho, H] = 0$. Hence, ρ has a spectral decomposition of the form

$$\rho = \sum_i \rho_i E_{ii}$$

Now, take $X = E_{ij}$ in the EEB inequalities. Then $[H, X] = (\lambda_i - \lambda_j)E_{ij}$, and the EEB inequality becomes:

$$\beta(\lambda_i - \lambda_j)\mathrm{Tr}\rho E_{jj} \geq f(\mathrm{Tr}\rho E_{jj}, \mathrm{Tr}\rho E_{ii})$$

By calculating the expectations this is

$$\beta(\lambda_i - \lambda_j)\rho_j \geq f(\rho_j, \rho_i) = \rho_j \log \frac{\rho_j}{\rho_i}. \tag{6.33}$$

We will first show that (6.33) implies that $\rho_i > 0$, for all i. Suppose that for some i, $\rho_i = 0$, and that $\rho_j > 0$ for some j, which must be the case since ρ is a density matrix. Then, the RHS of (6.33) equals $+\infty$, while the LHS is finite. We conclude that $\rho_i > 0$ for all i. Therefore we can divide both sides of (6.33) by ρ_j to obtain:

$$\beta(\lambda_i - \lambda_j) \geq \log \frac{\rho_j}{\rho_i}$$

By interchanging the roles of i and j in this inequality we see that the following equalities must hold for all i and j:

$$\beta(\lambda_i - \lambda_j) = \log \frac{\rho_j}{\rho_i}$$

or, equivalently

$$\rho_i = \text{constant} \times e^{-\beta\lambda_i},$$

where the constant is fixed by the normalization of ρ. This completes the proof that ρ_β is the only density matrix satisfying the EEB inequalities for a fixed H and $\beta \geq 0$. $\square$

7. Infinite Systems and the GNS representation

There is a close connection between representations and states of a C^*-algebra. The key to this connection is the so-called GNS construction, attributed to Israel Gelfand and Mark Naimark and, independently, Irving Segal, whose initials provided the name. The GNS constructions associates with each state ω on a C^*-algebra a canonical representation, which is unique up to unitary equivalence. We will discuss the GNS representation in its general setting first, and then apply it to the context of quantum spin systems. In particular,in the case of ground states of infinite quantum spin systems we obtain a representation in which the dynamics is generated by a densely defined self-adjoint operator (the GNS Hamiltonian) which is bounded below and of which the ground state is given by an eigenvector of the smallest eigenvalue.

7.1. **The GNS Construction.** Let $\mathcal{A}$ be a unital C^*-algebra. A *representation* of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a linear mapping $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ for which:
i) $\pi(\mathbb{1}) = \mathbb{1}$
ii) $\pi(A^*) = \pi(A)^*$
iii) $\pi(AB) = \pi(A)\pi(B)$.

All C^*-algebras appearing in this book are assumed to have an identity, generically denoted by $\mathbb{1}$, and all representations of a C^*-algebra on a Hilbert space will be assumed to map the identity into the identity operator on the Hilbert space.

A vector $\Omega \in \mathcal{H}$ is called *cyclic* for a representation π if

$$\mathcal{D}_\Omega = \{\pi(A)\Omega : A \in \mathcal{A}\} \subset \mathcal{H} \tag{7.1}$$

is a dense subspace of $\mathcal{H}$. A representation π is said to be *cyclic* if there is a cyclic vector for it.

Theorem 7.1 (GNS construction)**.** *Let ω be a state on a C^*-algebra $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$, and a vector $\Omega_\omega \in \mathcal{H}_\omega$, which is cyclic for π_ω and such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \quad \text{for all } A \in \mathcal{A}. \tag{7.2}$$

Moreover, the triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is uniquely determined by ω up to unitary equivalence. In other words, if there are two such cyclic representations $(\mathcal{H}_1, \pi_1, \Omega_1)$ and $(\mathcal{H}_2, \pi_2, \Omega_2)$ for the same state ω, then there exists a unitary mapping $U : \mathcal{H}_1 \to \mathcal{H}_2$ for which

$$\Omega_2 = U\Omega_1 \quad \text{and} \quad \pi_2(A) = U\pi_1(A)U^* \quad \text{for all } A \in \mathcal{A}. \tag{7.3}$$

The unitary U with these properties is itself unique.

Proof. First, we construct the Hilbert space. This is done in two steps. Note that

$$\langle A, B \rangle = \omega(A^*B) \tag{7.4}$$

defines a sesquilinear form on $\mathcal{A}$. Thus, $(\mathcal{A}, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space.

In general, this sesquilinear form may be degenerate; in which case it does not define an inner product. To remedy this, set

$$I = \{A \in \mathcal{A} : \omega(A^*A) = 0\}. \tag{7.5}$$

These correspond to the vectors with *zero norm.* One readily checks that I is a left ideal of $\mathcal{A}$. This means that I is a linear subspace of $\mathcal{A}$ for which given any $A \in \mathcal{A}$ and any $B \in I$, the product $AB \in I$. In words, I is invariant under left multiplication by elements in $\mathcal{A}$. To see this, we use $A^*A \leq \|A\|^2\mathbb{1}$. Since, for $B \in I$, $\omega(B^*B) = 0$, we then have

$$0 \leq \omega((AB)^*(AB)) = \omega(B^*A^*AB) \leq \omega(B^*B)\|A\|^2 = 0 \tag{7.6}$$

and thus $AB \in I$.

Let $\mathcal{H}_0$ be the quotient $\mathcal{A}/I$, which is the set of equivalence classes in $\mathcal{A}$ defined by I:

$$\psi_A = \{\tilde{A} \in \mathcal{A} : \tilde{A} = A + A' \text{ for some } A' \in I\} = A + I \tag{7.7}$$

As a quotient of a complex vector space by a subspace, $\mathcal{H}_0$ is a complex vector space, and we define a sesquilinear form in it by

$$\langle \psi_A, \psi_B \rangle = \langle A, B \rangle = \omega(A^* B) \tag{7.8}$$

for all equivalence classes ψ_A and ψ_B. It is clear that this form is well-defined: If $\tilde{A} = A + A'$ and $\tilde{B} = B + B'$ for some $A', B' \in I$, then

$$\omega((\tilde{A})^* \tilde{B}) = \omega(A^* B) + \omega(A^* B') + \omega((A')^* B) + \omega((A')^* B') = \omega(A^* B) \tag{7.9}$$

since the last three terms in the middle expression above vanish by Cauchy-Schwarz.

By construction, we now have a non-degenerate inner product space $\mathcal{H}_0$. We define $\mathcal{H}_\omega$ to be the Hilbert space obtained by completion of $\mathcal{H}_0$.

The next task is to define the representation π_ω of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H}_\omega)$. Observe that, again by construction, $\{\psi_A : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$. In this case, for any $A \in \mathcal{A}$, $\pi_\omega(A) \in \mathcal{B}(\mathcal{H}_\omega)$ will be uniquely defined if we define it on this dense subspace and show that

$$\|\pi_\omega(A)\| \leq \|A\|. \tag{7.10}$$

For any $A \in \mathcal{A}$, define a mapping $\pi_\omega(A)$ on $\mathcal{H}_0$ by setting

$$\pi_\omega(A)\psi_B = \psi_{AB} \tag{7.11}$$

This is well-defined because I is a left-ideal:

$$\pi_\omega(A)\psi_{B+B'} = \psi_{AB+AB'} = \psi_{AB} \quad \text{since } AB' \in I. \tag{7.12}$$

Since

$$\alpha\psi_C + \beta\psi_D = \psi_{\alpha C + \beta D}\,, \tag{7.13}$$

linearity of $\pi_\omega(A)$ is also easy to check. Moreover,

$$\|\pi_\omega(A)\psi_B\|^2 = \langle \psi_{AB}, \psi_{AB} \rangle = \omega(B^* A^* A B) \leq \|A\|^2 \omega(B^* B) = \|A\|^2 \|\psi_B\|^2 \tag{7.14}$$

which proves that $\|\pi_\omega(A)\| \leq \|A\|$ on $\mathcal{H}_0$. We will continue to denote by $\pi_\omega(A)$ the unique bounded extension of this mapping to all of $\mathcal{H}_\omega$. One easily checks that π_ω is a representation:

$$\begin{aligned}
&\pi_\omega(\mathbb{1})\psi_B = \psi_B \\
&\langle \pi_\omega(A^*)\psi_B, \psi_C \rangle = \omega((A^* B)^* C) = \omega(B^* A C) = \langle \psi_B, \pi_\omega(A)\psi_C \rangle \\
&\pi_\omega(A_1 A_2)\psi_B = \psi_{A_1 A_2 B} = \pi_\omega(A_1)\psi_{A_2 B} = \pi_\omega(A_1)\pi_\omega(A_2)\psi_B.
\end{aligned}$$

Finally, $\psi_{\mathbb{1}}$ is a cyclic vector with the desired properties. It is cyclic because

$$\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\} = \{\psi_A : A \in \mathcal{A}\} \tag{7.15}$$

is dense by construction of the Hilbert space. The desired relation with the state ω follows from the definition of the inner product on $\mathcal{H}_\omega$:

$$\langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle = \langle \psi_{\mathbb{1}}, \psi_A \rangle = \omega(A). \tag{7.16}$$

This completes the construction of a GNS triple $(\mathcal{H}_\omega, \pi_\omega, \Omega)$.

To prove the uniqueness statement, suppose there are two GNS triples, $(\mathcal{H}_1, \pi_1, \Omega_1)$ and $(\mathcal{H}_2, \pi_2, \Omega_2)$, with

$$\langle \Omega_1, \pi_1(A)\Omega_1 \rangle = \omega(A) = \langle \Omega_2, \pi_2(A)\Omega_2 \rangle \quad \text{for all } A \in \mathcal{A} \tag{7.17}$$

In this case, define a mapping $U : \mathcal{H}_1 \to \mathcal{H}_2$ by setting

$$U\pi_1(A)\Omega_1 = \pi_2(A)\Omega_2 \quad \text{for all } A \in \mathcal{A}. \tag{7.18}$$

Since Ω_1 is cyclic, and due to (7.17), this defines an isometry U on a dense set of vectors in $\mathcal{H}_1$; we will continue to denote by U the unique linear extension of this mapping to all of $\mathcal{H}_1$. Observe that

$$\begin{aligned}\langle U\pi_1(A)\Omega_1, U\pi_1(B)\Omega_1\rangle &= \langle \pi_2(A)\Omega_2, \pi_2(B)\Omega_2\rangle \\ &= \langle \Omega_2, \pi_2(A^*B)\omega_2\rangle \\ &= \omega(A^*B) = \langle \pi_1(A)\Omega_1, \pi_1(B)\Omega_1\rangle.\end{aligned} \tag{7.19}$$

It follows that U is unitary, i.e. $U^*U = UU^* = \mathbb{1}$. It is straightforward to check the remaining properties of U. □

The uniqueness property of the GNS triple for a given state ω, which we henceforth denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, has an important consequence.

Corollary 7.2. *Let $\mathcal{A}$ be a C^*-algebra, ω a state on $\mathcal{A}$ and α an automorphism of $\mathcal{A}$ that leaves ω invariant, i.e. , such that $\omega \circ \alpha = \omega$. Then, α is implementable in the GNS representation of ω. Explicitly, there exists a unique unitary $U \in \mathcal{B}(\mathcal{H}_\omega)$ such that for all $A \in \mathcal{A}$ we have*

$$\pi_\omega(\alpha(A)) = U\pi_\omega(A)U^*. \tag{7.20}$$

Proof. If $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is a GNS triple for ω, one can immediately check that $(\mathcal{H}_\omega, \pi_\omega \circ \alpha, \Omega_\omega)$ is a GNS triple for $\omega \circ \alpha$. By Theorem 7.1, the existence and uniqueness of a unitary operator U on $\mathcal{H}_\omega$ with the desired properties is then guaranteed:

$$U\Omega_\omega = \Omega_\omega, \quad \pi_\omega(\alpha(A)) = U\pi_\omega(A)U^*. \tag{7.21}$$

□

As a first application of this corollary, we explore its consequences for ground states, which are invariant under the automorphisms τ_t describing the time evolution (see Proposition 6.3).

Let ω be a ground state. From Corollary 7.2 we then have, for each $t \in \mathbb{R}$, a unique unitary U_t on $\mathcal{H}_\omega$ implementing τ_t (for consistency with the standard conventions in the quantum mechanics literature, we interchange the roles of U_t and U_t^*. Concretely, we have

$$U_t\Omega_\omega = \Omega_\omega \quad \text{and} \quad U_t^*\pi_\omega(A)U_t = \pi_\omega(\tau_t(A)) \quad \text{for all } A \in \mathcal{A}_\Gamma. \tag{7.22}$$

Using the uniqueness, one sees that the group property of the τ_t carries over to the U_t. and the continuity for the norm on $\mathcal{A}$ of $t \mapsto \tau_t(A)$ implies that $t \mapsto U_t\psi$ is continuous for all $\psi \in \mathcal{H}_\omega$. This is the strong continuity property of the one-parameter group of unitaries. To verify this, consider, for all $A \in \mathcal{A}$,

$$\begin{aligned}\|U_t^*\pi_\omega(A)\Omega_\omega - \pi_\omega(A)\Omega_\omega\| &= \|U_t^*\pi_\omega(A)U_tU_t^*\Omega_\omega - \pi_\omega(A)\Omega_\omega\| \\ &= \|\pi_\omega(\tau_t(A))\Omega_\omega - \pi_\omega(A)\Omega_\omega\| \\ &= \|\pi_\omega(\tau_t(A) - A)\Omega_\omega\|.\end{aligned} \tag{7.23}$$

The last quantity vanishes as $t \to 0$ by the strong continuity of τ_t.

Using Stone's Theorem 7.3, we now conclude that there is a densely defined self-adjoint operator H_ω acting on (a dense subset of) $\mathcal{H}_\omega$ for which

$$U_t = e^{-itH_\omega}. \tag{7.24}$$

We are now back to quantum mechanics on a Hilbert space, i.e., the Schrödinger picture in which one studies self- adjoint operators on a Hilbert space.

Note that using

$$U_t\Omega_\omega = \Omega_\omega, \tag{7.25}$$

and the ground state property of ω, we have that

$$\begin{aligned} 0 \leq \omega(A^*\delta(A)) &= \frac{1}{i}\frac{d}{dt}\omega(A^*\tau_t(A))|_{t=0} \\ &= \frac{1}{i}\frac{d}{dt}\langle \Omega_\omega, \pi_\omega(A^*\tau_t(A))\Omega_\omega\rangle|_{t=0} \\ &= \frac{1}{i}\frac{d}{dt}\langle \pi_\omega(A)\Omega_\omega, U_t^*\pi_\omega(A)U_t\Omega_\omega\rangle|_{t=0} \\ &= \langle \pi_\omega(A)\Omega_\omega, H_\omega\pi_\omega(A)\Omega_\omega\rangle \end{aligned} \tag{7.26}$$

and hence $H_\omega \geq 0$. This means that in its GNS representation a ground state is represented by an eigenvector of the Hamiltonian H_ω with eigenvalue 0, which corresponds to $\inf \mathrm{spec} H_\omega$.

For the readers's convenience, we include a statement of Stone's theorem here. See, e.g., [16] or [18] for a proof.

Theorem 7.3 (Stone's Theorem)**.** *For all $t \in \mathbb{R}$, let U_t be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then, $\{U_t \mid t \in \mathbb{R}\}$ is a strongly continuous one-parameter group of unitary operators if and only if there exists a densely defined self-adjoint operator H with domain $D(H)$ such that*

$$U_t = e^{-itH}. \tag{7.27}$$

Moreover,

$$D(H) = \{\psi \in \mathcal{H} \mid \exists \phi \in \mathcal{H} \text{ such that } \lim_{t\to 0} \|t^{-1}(e^{-itH}\psi - \psi) - \phi\| = 0\}. \tag{7.28}$$

7.2. **Ground States and Equilibrium States for Infinite Systems.** Let us summarize the basic mathematical elements describing an infinite (or finite) quantum spin system. Associated with the points of a countable metric space (Γ, d), often referred to as the "lattice", are finite-dimensional complex Hilbert spaces $\mathcal{H}_x$. We introduced the algebra of local observables $\mathcal{A}_\Gamma^{\mathrm{loc}}$ and the C^*-algebra of quasi-local observables, $\mathcal{A}_\Gamma$, which is the norm completion of $\mathcal{A}_\Gamma^{\mathrm{loc}}$. We defined spaces, denoted by $\mathcal{B}_F(\Gamma$, of interactions $\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma^{\mathrm{loc}}$, in terms of the so-called F-norms $\|\cdot\|_F$. For each such Φ there is a strongly continuous one-parameter group of automorphisms of $\mathcal{A}_\Gamma$, $\{\tau_t \mid t \in \mathbb{R}\}$, describing the dynamics of the system. There is a densely defined generator δ, for which $\mathcal{A}_\Gamma^{\mathrm{loc}}$ is a core, and such that $\tau_t = e^{it\delta}$.

Equilibrium states at inverse temperature $\beta \in [0, \infty)$, and ground states ($\beta = \infty$), can be characterized in terms of δ as follows. Let $\mathcal{S}_\beta$ be the set of states on $\mathcal{A}$ that satisfy the following conditions:

i. For $\beta = \infty$, the set of ground states $\mathcal{S}_\infty$ consists of those states ω such that

$$\omega(A^*\delta(A)) \geq 0, \quad \text{for all } A \in \mathcal{A}_{\mathrm{loc}}; \tag{7.29}$$

ii. For $\beta \in (0, \infty)$, the set of equilibium states at inverse temperature β, $\mathcal{S}_\beta$ consists of those states ω such that

$$\omega(A^*\delta(A)) \geq \frac{1}{\beta}\omega(A^*A) \ln\left[\frac{\omega(A^*A)}{\omega(AA^*)}\right], \quad \text{for all } A \in \mathcal{A}_{\mathrm{loc}}; \tag{7.30}$$

iii. For $\beta = 0$, describing infinite temperature, we have a unique state ω_0 such that

$$\omega_0(A^*A) = \omega_0(AA^*), \quad \text{for all } A \in \mathcal{A}_{\mathrm{loc}}. \tag{7.31}$$

Note that ω_0 does not depend on δ. It is called the tracial state and, in fact, satisfies $\omega_0(AB) = \omega(BA)$, for all $A, B \in \mathcal{A}_\Gamma$.

One can show that for all $\beta \in [0, \infty]$ the set $\mathcal{S}_\beta$ is a convex subset of the set of all states on $\mathcal{A}$, meaning that if $\omega_1, \omega_2 \in \mathcal{S}_\beta$, then $t\omega_1 + (1-t)\omega_2 \in \mathcal{S}_\beta$ for all $t \in [0, 1]$.

The following bit of terminology about convex sets will be useful.

Let C be a convex subset of a real vector space V. The point $c \in C$ is said to be an extreme point of C if it is not a non-trivial convex combination of points in C, i.e. if $c_1, c_2 \in C$ and there is some $t \in (0,1)$ for which $c = tc_1 + (1-t)c_2$, then $c_1 = c_2 = c$. We will denote by $\mathcal{E}(C)$ the set of all extreme points in C.

If $c_1, c_2 \in C$ then the *segment* between c_1 and c_2 is the set

$$\{tc_1 + (1-t)c_2 : t \in [0,1]\} \subset C \tag{7.32}$$

C is said to be a *simplex* if for all $c \in C$, there exists a unique set of points $c_i \in \mathcal{E}(C)$, $i = 1, \ldots, n$, and $t_i \in (0,1)$ with $\sum_i t_i = 1$, such that $\sum_{i=1}^n t_i c_i = c$. Strictly speaking, this only defines finite-dimensional simplices. Working with infinite-dimensional simplices involves a choice of topology and possibly measure theory to generalize to convex combinations of infinite sets of points.

A set $F \subset C$ is said to be a *face* if it is convex and whenever $f \in F$ and $f = tc_1 + (1-t)c_2$ for some $c_1, c_2 \in C$ and $t \in (0,1)$, then $tc_1 + (1-t)c_2 \in F$ for all $t \in [0,1]$.

The set of all ground states of a given dynamics, $\mathcal{S}_\infty$, is a face in the set of all states. The extreme points of $\mathcal{S}_\infty$ are pure states, i.e., they are also extreme points in the set of all states.

In contrast, the set of all equilibrium states for a system at $\beta \in (0,\infty)$, $\mathcal{S}_\beta$, is a simplex in the set of all states. The extreme points are factor states, a notion we define in the next paragraph. They are not pure states.

Let ω be a state on a C^*-algebra $\mathcal{A}$. Consider the GNS triple associated to ω, namely $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Recall that

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \quad \text{for all } A \in \mathcal{A} \tag{7.33}$$

and $\pi_\omega(\mathcal{A}) \subset \mathcal{B}(\mathcal{H}_\omega)$ is a sub-algebra.

The commutant of $\pi_\omega(\mathcal{A})$ is defined by

$$\pi_\omega(\mathcal{A})' = \{B \in \mathcal{B}(\mathcal{H}_\omega) : [B, \pi_\omega(A)] = 0 \text{ for all } A \in \mathcal{A}\}. \tag{7.34}$$

The bi-commutant is then

$$\pi_\omega(\mathcal{A})'' = \left(\pi_\omega(\mathcal{A})'\right)'. \tag{7.35}$$

A famous theorem of von Neumann's states that

$$\pi_\omega(\mathcal{A})'' = \overline{\pi_\omega(\mathcal{A})}^{\mathrm{w}}, \tag{7.36}$$

i.e. the bi-commutant of $\pi_\omega(\mathcal{A})$ is the closure of $\pi_\omega(\mathcal{A})$ in the weak operator topology on $\mathcal{B}(\mathcal{H}_\omega)$.

The *center* of a representation π is:

$$Z_\pi = \pi(\mathcal{A})' \cap \pi(\mathcal{A})'' \tag{7.37}$$

A state ω is said to be a *factor state* (also know as a *primary state*) if and only if

$$Z_{\pi_\omega} = \pi_\omega(\mathcal{A})' \cap \pi_\omega(\mathcal{A})'' = \mathbb{C}\mathbb{1} \tag{7.38}$$

The GNS representation of a pure state ω is *irreducible*, meaning that $\pi_\omega(\mathcal{A})' = \mathbb{C}\mathbb{1}$. It follows that $\pi_\omega(\mathcal{A})'' = \mathcal{B}(\mathcal{H}_\omega)$.

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Exercises

Exercise 1. Suppose $\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma^{\rm loc}$ is an interaction such that $\Phi(\emptyset) = 0$ and $\|\Phi\|_F < \infty$, for some F-function F, and define $\tilde{\Phi}(X) = |X|^{-1}\Phi(X)$, for all $\emptyset \neq X \in \mathcal{P}_0(\Gamma)$, and $\tilde{\Phi}(\emptyset) = 0$. Prove the inequality

$$\sup_{x\in\Gamma} \sum_{\substack{X\in\mathcal{P}_0(\Gamma)\\ x\in X}} \|\Phi(X)\| \leq \|F\| \|\tilde{\Phi}\|_F.$$

Exercise 2. Show that for any $\epsilon > 0$, $\nu = 1, 2, \ldots$, F defined by

$$F(r) = (1+r)^{-(\nu+\epsilon)} \tag{7.39}$$

is an F-function for $\Gamma = \mathbb{Z}^\nu$, with constant

$$C = 2^{\nu+\epsilon}\|F\|. \tag{7.40}$$

Exercise 3. Suppose $\Phi \in \mathcal{B}_F(\Gamma)$ and define, for $\Lambda \in \mathcal{P}_0(\Gamma)$,

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X).$$

Show that for each increasing sequence $\Lambda_n \in \mathcal{P}_0(\Gamma)$, such that $\bigcup_n \Lambda_n = \Gamma$, and $A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$, the sequence

$$\delta_{\Lambda_n}(A) = [H_{\Lambda_n}, A]$$

is a Cauchy sequence in $\mathcal{A}_\Gamma$, and converges to a limit independent of the chosen sequence (Λ_n).

Exercise 4. Give a proof of Proposition 6.3

Exercise 5. Give a proof of Proposition 6.7

Exercise 6. Let ω a state on the C^*-algebra M_d of all $d \times d$ complex matrices and denote by ρ the density matrix such that $\omega(A) = \mathrm{Tr}\rho A$.
a) Suppose $\ker\rho = \{0\}$ and let $e_1, \ldots, e_d$ denote an orthonormal basis of eigenvectors of ρ corresponding to its eigenvalues $\rho_1, \ldots, \rho_d$. Define $\mathcal{H}_\omega = \mathbb{C}^d \otimes \mathbb{C}^d$, $\pi_\omega(A) = A \otimes \mathbb{1}$, and $\Omega_\omega = \sum_{i=1}^d \sqrt{\rho_i} e_i \otimes e_i$. Prove that $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is the GNS triple of ω.
b) Find the commutant $\pi_\omega(\mathcal{A})'$, and the bi-commutant $\pi_\omega(\mathcal{A})''$.
c) What is the GNS representation if ω is a pure state on M_d?

Exercise 7. Let $\mathcal{A}_\Gamma$ be the C^*-algebra of quasi-local observables of a quantum spin system. Show that there is a unique tracial state on $\mathcal{A}_\Gamma$, i.e., a unique state ω such that $\omega(AB) = \omega(BA)$, for all $A, B \in \mathcal{A}_\Gamma$.