TECHNISCHE UNIVERSITÄT MÜNCHEN

Zentrum Mathematik

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Mathematical Foundations of Machine Learning

http://www-m5.ma.tum.de/Allgemeines/MA4801_2018S

Summer 2018
Blatt 3
(2.05.2018)

The following homework will be discussed on May 8th and 11th:

H3.1. (Covering and Packing Numbers)

a) Show that for any norm on $\mathbb{R}^d$, the $\epsilon-$packing number $M(B_r, \epsilon)$ for the norm ball $B_r := \{x \in \mathbb{R}^d \,|\, \|x\| \leq r\}$ satisfies $M(B_r, \epsilon) \geq \left(\frac{r}{\epsilon}\right)^d$.

b) Let $(\mathcal{M}, d)$ be metric space, $A \subset \mathcal{M}$ and let $\beta(A, \epsilon) \in \mathbb{N}$ the smallest number of bits sufficient to specify every $a \in A$ up to an error at most ϵ. Prove that

$$\lceil \log_2 M(A, \epsilon) \rceil \geq \beta(A, \epsilon) \geq \log_2 N(A, \epsilon), \tag{1}$$

where $N(A, \epsilon)$ is the $\epsilon-$covering number, $M(A, \epsilon)$ the $\epsilon-$packing number and $\lceil \cdot \rceil$ the ceiling function.

H3.2. (Pseudo-dimension)

a) Let $\mathcal{F} \subset \mathbb{R}^{\mathcal{X}}$ be a real vector space of functions of dimension d. Prove that $P\dim(\mathcal{F}) = d$.

b) Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function, $\mathcal{G} \subset \mathbb{R}^{\mathcal{X}}$ and $\sigma \circ \mathcal{G} := \{\sigma \circ g | g \in \mathcal{G}\}$. Show that $P\dim(\sigma \circ \mathcal{G}) \leq P\dim(\mathcal{G})$.

H3.3. (Fat-Shattering dimension)

a) Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be the set of monic polynomials of degree k. Prove a finite upper bound on the $\epsilon-$fat-shattering dimension.

b) Let $\mathcal{X} := \{x | \, \|x\| \leq 1\}$ be the unit ball of an infinite dimensional Hilbert space and $\mathcal{F} := \{\mathcal{X} \ni x \mapsto \langle x, f \rangle | \, \|f\| \leq 1\}$. Show that the $\epsilon-$fat-shattering dimension satisfies $fat(\mathcal{F}, \epsilon) \geq \lfloor 4\epsilon^{-2} \rfloor$. Here $\lfloor \cdot \rfloor$ denotes the floor function.