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Mathematical Foundations of Machine Learning

http://www-m5.ma.tum.de/Allgemeines/MA4801_2018S

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The following homework will be discussed on April 24th and 27th:

H2.1. Read section 1.3, 1.5 and 1.6 in the lecture notes.

H2.2. (PAC Learning of threshold functions) Consider the set of all threshold functions $\mathcal{F} \subset \{-1,1\}^{\mathbb{R}}$ defined by $\mathcal{F} := \{x \mapsto \operatorname{sgn}[x-b]\}_{b\in\mathbb{R}}$. Let us take as Risk function R the error probability and $(\epsilon, \delta) \in (0,1]^2$. We would like to learn a threshold function $\hat{c} \in \mathcal{F}$ and so sample randomly n times over $\mathcal{X} = \mathbb{R}$ and collect the outcomes of $\hat{c}$ in $S := ((x_1, \hat{c}(x_1)), \ldots, (x_n, \hat{c}(x_n)))$. Furthermore, assume that the output of the learning algorithm returns a hypothesis $h_S \in \mathcal{F}$ for which $\hat{R}(h_S) = 0$. Show that $R(h) \leq \epsilon$ holds with probability at least $1-\delta$ over repeated sampling of training set of size n if

$$n \geq \frac{1}{\epsilon} \log \frac{2}{\delta}.$$

H2.3. Consider function classes $\mathcal{F}_1, \mathcal{F}_2 \subset \{-1,1\}^{\mathcal{X}}$ with finite VC-dimensions. Show that

$$\operatorname{VCdim}(\mathcal{F}_1 \cup \mathcal{F}_2) \leq \operatorname{VCdim}(\mathcal{F}_1) + \operatorname{VCdim}(\mathcal{F}_2) + 1.$$

H2.4. (Symmetric rectangles) Let $\mathcal{C} := \big\{A \subset \mathbb{R}^n \,|\, \exists a \in \mathbb{R}^n_+ : A = [-a_1, a_1] \times \ldots \times [-a_n, a_n]\big\}$ be the set of symmetric rectangles in $\mathbb{R}^n$ and $\mathcal{F} := \{f : \mathbb{R}^n \to \{0,1\} | \exists C \in \mathcal{C} : f(x) = \mathbb{1}_{x\in C}\}$ the corresponding class of indicator functions. Compute the $VC-$dimension of $\mathcal{F}$.

H2.5. (Closed balls) Let $\mathcal{B} := \{B \subset \mathbb{R}^n \,|\, \exists (v,r) \in \mathbb{R}^n \times \mathbb{R}_+ : x \in B \Leftrightarrow \|x-v\|_2 \leq r\}$ be the class of all closed balls in $\mathbb{R}^n$ and $\mathcal{F} := \{f : \mathbb{R}^n \to \{0,1\} | \exists B \in \mathcal{B} : f(x) = \mathbb{1}_{x\in B}\}$ the corresponding class of indicator functions. Show that the $\operatorname{VCdim}(\mathcal{F}) \leq n+2$.