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Mathematical Foundations of Machine Learning

http://www-m5.ma.tum.de/Allgemeines/MA4801_2018S

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Blatt 1
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The following homework will be discussed on April 17th:

H1.1. Read introduction, section 1.1 and section 1.2 in the lecture notes.

H1.2. (Basic Inequalities) Let $h : \mathbb{R} \to [0, \infty)$ be a nonnegative function and let X be a real-valued random variable. Show that

$$\mathbb{P}\left[h(X) \geq a\right] \leq \frac{\mathbb{E}\left[h(X)\right]}{a} \qquad \text{for all } a > 0.$$

Use this to obtain

a) Markov's inequality:

$$\mathbb{P}\left[\, |X| \geq a \,\right] \leq \frac{\mathbb{E}\left[\, |X| \,\right]}{a} \qquad \text{for all } a > 0,$$

b) Chebyshev's inequality:

$$\mathbb{P}\left[\, |X - \mathbb{E}\left[X\right]| \geq a \,\right] \leq \frac{\mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right]}{a^2} \qquad \text{for all } a > 0.$$

H1.3. (Weak-Law of Large Numbers) Let (X_j) be a sequence of independent identically distributed (i.i.d in short) real random variables with finite mean and variance. Show that

$$\mathbb{P}\left[\left|\sum_{j=1}^{n} \frac{X_j}{n} - \mathbb{E}\left[X\right]\right| > \varepsilon\right] \longrightarrow 0 \quad \text{as } n \to \infty.$$

H1.4. (Hoeffding Inequality) Let $Z_1, \ldots, Z_n$ be real independent random variables whose values are contained in the intervals $[a_i, b_i] \supseteq Z_i$. Show that for every $\epsilon > 0$ it holds that

$$\mathbb{P}\left[\sum_{i=1}^{n} Z_i - \mathbb{E}\left[Z_i\right] \geq \epsilon\right] \leq \exp\left[-\frac{2\epsilon^2}{\sum_{i=1}^{n}(a_i - b_i)^2}\right].$$

H1.5. A medical research group wants to introduce a new treatment for a difficult illness, which could in their eyes be very profitable as a business. In order to justify the effectiveness of the treatment they will of course test it. The cost of performing the test outside the country is way too expensive. However, they plan to advertise the new treatment all over the world and so decide to test it in Hospitals abroad, but with less frequency. They provide with probability 91% the new treatment to inland patients and the remaining 9% receives the standard treatment; for abroad patients only 1% of them receives the new treatment and the rest the standard treatment. The results are summarized in the following tables:

	Inland Patients Only		Abroad Patients Only	
Treatment	Standard	New	Standard	New
Dead	950	9000	5000	5
Alive	50	1000	5000	95

Treatment	Standard	New
Dead	5950	9005
Alive	5050	1095

Let $\mathbb{P}[A|B]$ be the probability of ocurring the event A given that B has happened (or that B is true).

a) Compute $\mathbb{P}[\text{Patient is Alive} \mid \text{Inland patient, new treatment used}]$ and $\mathbb{P}[\text{Patient is Alive} \mid \text{Inland patient, standard treatment used}]$.

b) Compute $\mathbb{P}[\text{Patient is Alive} \mid \text{Abroad patient, new treatment used}]$ and $\mathbb{P}[\text{Patient is Alive} \mid \text{Abroad patient, standard treatment used}]$.

c) Based on these results can this medical group argue in favour of the effectiveness of the new treatment and be right? Note that the second table is obtained from the first one.

d) (Optional) Compare $\mathbb{P}[\text{Patient is Alive} \mid \text{Inland patient}]$ and $\mathbb{P}[\text{Patient is Alive} \mid \text{Patient Abroad}]$.