For $\mathcal{G} \subseteq \mathbb{R}^{\mathcal{X}}$ and $x \in \mathcal{X}^n$ define seminorm $\|g\|_{p,x} := \left(\frac{1}{n} \sum_{i=1}^{n} |g(x_i)|^p \right)^{1/p}$ for $p \in [1,\infty)$

and $\|g\|_{\infty,x} := \max_i |g(x_i)|$ on span $\mathcal{G}$.

Note that $M(\varepsilon, \mathcal{G}, \|\cdot\|_{p,z}) \leq M(\varepsilon, \mathcal{G}, \|\cdot\|_{q,z})$ if $p \leq q$ and similar for the covering numbers.

Note for $\mathcal{F} \subseteq \{0,1\}^{\mathcal{X}}$: $\|f-g\|_{1,x} = \mathbb{P}[f(X) = g(X)]$ if an element x_i is taken uniformly at random from x.

Moreover, $\|f-g\|_{1,x} = \|f-g\|_{p,x}^p$ so that $N(\varepsilon, \mathcal{F}, \|\cdot\|_{p,x}) = N(\varepsilon^p, \mathcal{F}, \|\cdot\|_{1,x})$

Since $\|g-f\|_{\infty,x} = \max_i |g(x_i) - f(x_i)| = 1$ iff $f|_x \neq g|_x$

we have $N_{in}(\varepsilon, \mathcal{F}, \|\cdot\|_{\infty,x}) = |\mathcal{F}|_x|$ if $\varepsilon < 1$.

Def.: "uniform covering number" for $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{X}}$:

$$\Gamma_p(\varepsilon, \mathcal{F}, n) := \max_{A \in \mathcal{X}^n} N_{in}(\varepsilon, \mathcal{F}, \|\cdot\|_{p,A})$$

Then $\Gamma_\infty(\varepsilon, \mathcal{F}, n) = \Gamma(n)$ if $\varepsilon < 1$ and $\mathcal{F} \in \{0,1\}^{\mathcal{X}}$.

Prop.: $\mathcal{F} \subseteq \mathcal{Y}^{\mathcal{X}}$, $L: \mathcal{Y} \times \mathcal{Y} \to [0,c]$, $\varepsilon > 0$. For any prob. measure P on $\mathcal{X} \times \mathcal{Y}$:

$$\mathbb{P}_{S \sim P^n}\left[\exists h \in \mathcal{F} : |R(h) - \hat{R}(h)| \geq \varepsilon \right] \leq 4\, \Gamma_1\left(\frac{\varepsilon}{8}, \mathcal{G}, 2n \right) e^{-\frac{n\varepsilon^2}{32 c^2}}$$

where $\mathcal{G} := \{ g : \mathcal{X} \times \mathcal{Y} \to [0,c] \mid \exists h \in \mathcal{F} : g(x,y) = L(y, h(x)) \}$.

proof: ... lecture notes. $\square$

Def.: The pseudo-dimension of $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{X}}$ is defined as

$$\mathrm{Pdim}(\mathcal{F}) := \mathrm{VCdim}\{\, \mathcal{X}\times\mathbb{R} \ni (x,y) \mapsto \mathrm{sgn}[f(x)-y] \mid f \in \mathcal{F} \}$$

If $\mathrm{Pdim}(\mathcal{F}) \geq d$, then there is $\{(x_i,y_i)\}_{i=1}^{d} \subseteq \mathcal{X}\times\mathbb{R}$ s.t. for all subsets $C \subseteq \{1,\dots,d\}$ there is a function $f \in \mathcal{F}$ s.t. $f(x_i) \geq y_i \Leftrightarrow i \in C$. $\mathrm{Pdim}(\mathcal{F})$ is the largest such d.

Def.: Let $\alpha \in (0,\infty)$. The α-fat-shattering dimension of $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{X}}$ is the largest $d \in \mathbb{N} \cup \{\infty\}$ for which there exist $\{(x_i,y_i)\}_{i=1}^{d}$ s.t. for all subsets $C \subseteq \{1,\dots,d\}$ there is a function $f \in \mathcal{F}$ s.t.

$$f(x_i) \geq y_i + \frac{\alpha}{2} \quad \text{if } i \in C$$
$$f(x_i) \leq y_i - \frac{\alpha}{2} \quad \text{if } i \notin C$$

Cor.: Let $\mathcal{F}' \subseteq \mathcal{F} \subseteq \mathbb{R}^{\mathcal{X}}$, $\alpha > \alpha' > 0$ and $\mathrm{sgn}[\mathcal{F}] := \{ x \mapsto \mathrm{sgn}[f(x)] \mid f \in \mathcal{F}\}$. Then

(i) $\mathrm{Pdim}(\mathcal{F}) \geq \mathrm{Pdim}(\mathrm{sgn}[\mathcal{F}]) = \mathrm{VCdim}(\mathrm{sgn}[\mathcal{F}])$

(ii) $\mathrm{Pdim}(\mathcal{F}) \geq \mathrm{Pdim}(\mathcal{F}')$ and $\mathrm{fat}(\mathcal{F},\alpha) \geq \mathrm{fat}(\mathcal{F}',\alpha)$

(iii) $\mathrm{fat}(\mathcal{F},\alpha) \leq \mathrm{fat}(\mathcal{F},\alpha') \leq \mathrm{Pdim}(\mathcal{F})$

(iv) $\lim_{\alpha\to 0} \mathrm{fat}(\mathcal{F},\alpha) = \mathrm{Pdim}(\mathcal{F})$

(v) If $\mathcal{F}$ is closed under scalar multiplication, then $\mathrm{fat}(\mathcal{F},\alpha) = \mathrm{Pdim}(\mathcal{F})$.

Prop.: If $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{X}}$ is a real vector space of dimension d, then $\mathrm{Pdim}(\mathcal{F}) = \mathrm{fat}(\mathcal{F},\alpha) = d$ for all $\alpha > 0$.

proof: $\mathrm{Pdim}(\mathcal{F}) = \mathrm{VCdim}\{ (x,y) \mapsto \mathrm{sgn}[f(x)-y] \mid f \in \mathcal{F} \}$

Use $\mathcal{G} := \{ (x,y) \mapsto \mathcal{F}(x) \}$, $\varphi(x,y) := y$. $\square$

example Let $\mathcal{F}$ be the class of real polynomials on $\mathbb{R}^k$ of degree d. Since this is a vector space of dim $\binom{k+d}{d}$ we have $\mathrm{Pdim}(\mathcal{F}) = \binom{k+d}{d}$.