The NLTS conjecture

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Abstract

These notes serve as the basis of a talk given in the context of a research seminar on open problems in quantum information theory (`https://www-m5.ma.tum.de/Allgemeines/HS_ResearchSemQIT2019`). The goal of this talk is to give a brief introduction to the NLTS conjecture. We first motivate NLTS as a natural extension of the concept of topological order before stating the conjecture precisely. After that, we highlight its importance for the existence of a quantum analog to the celebrated classical PCP theorem. Finally, we discuss some approaches to resolve variants of the conjecture.

1 Motivation, Setup and Statement

The central object of interest in this talk will be so-called local Hamiltonians:

Definition 1.1 (Local Hamiltonians)
*Let $\mathcal{I} \subset \mathbb{N}$ be an index set. A **family of local Hamiltonians** is a set of Hamiltonians $\{H^{(n)}\}_{n\in\mathcal{I}}$, where each $H^{(n)}$ is defined on n finite-dimensional subsystems (in the following taken to be qubits), that are of the form*

$$H^{(n)} = \sum_m H_m^{(n)}$$

where each $H_m^{(n)}$ acts non-trivially on $O(1)$ qubits. We furthermore require that the operator norm of $H_m^{(n)}$ is bounded by a constant independent of n and that each qubit is only involved in a constant number of terms $H_m^{(n)}$.

1.1 Topological Order

Of key interest concerning local Hamiltonians is the concept of topological order. There are several ways to define this notion; we will use the definition based on so-called "non-trivial ground states":

Definition 1.2 (Topological Order)
*Let $\mathcal{I} \subset \mathbb{N}$ be an infinite set of system sizes. A family of local gapped Hamiltonians $\{H^{(n)}\}_{n\in\mathcal{I}}$ is called **topologically ordered** if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that*

- *for all $n \in \mathcal{I}$, $H^{(n)}$ has ground energy 0*
- *$\langle 0^n | U^\dagger H^{(n)} U | 0^n \rangle > 0$ for any depth-d circuit U consisting of two qubit gates and for any $n \in \mathcal{I}$ with $n \geq f(d)$.*

In other words, a family of local gapped Hamiltonians is called topologically ordered if any ground states cannot be prepared from a product state by a constant-depth circuit.

Example 1.3 (Toric code)
It is known that preparing a ground state of Kitaev's toric code [Kit03] from a product state requires a circuit depth at least

- *polynomially in the system size using nearest-neighbor gates [BHV06]*
- *logarithmically using non-local gates [AT18]*

A natural question to ask next is whether there exists a family of Hamiltonians for which this property does not only hold with regard to its ground states, but also for (averaged) low-energy states. This leads us to the NLTS property.

1.2 NLTS - "No Low-Energy Trivial States"

Definition 1.4 (NLTS)
*Let $\mathcal{I}$ be an infinite set of system sizes. A family of local Hamiltonians $\{H^{(n)}\}_{n\in\mathcal{I}}$ has the **NLTS property** if there exists $\varepsilon > 0$ and a function $f\colon \mathbb{N} \to \mathbb{N}$ such that*

- *for all $n \in \mathcal{I}$, $H^{(n)}$ has ground energy 0*
- *$\langle 0^n | U^\dagger H^{(n)} U | 0^n \rangle > \varepsilon n$ for any depth-d circuit U consisting of two qubit gates and for any $n \in \mathcal{I}$ with $n \geq f(d)$.*

Unlike in the "$\varepsilon = 0$" case of Definition 1.2 where we could give the example of the toric code, we do not know whether such a family exists:

Conjecture 1 (NLTS conjecture, [FH14])
There exists a family of local Hamiltonians with the NLTS property.

2 Importance of NLTS: the qPCP conjecture

In this section, we will discuss the main motivation for the NLTS conjecture: it can be thought of as a precursor to a qPCP theorem, meaning that the NLTS conjecture is necessary for a potential qPCP theorem to hold.

2.1 The PCP theorem

Before we can talk about the qPCP conjecture, we should take a look at the classical analog which is already known to be true:

Theorem 2.1 (PCP theorem - hardness of approximation)
There is a constant $\varepsilon > 0$ such that, given a $3SAT$ formula ϕ, it is NP*-hard to decide which of the following is true (if promised that one of these holds):*

- *ϕ is satisfiable*
- *at most a fraction of $(1 - \varepsilon)$ of the clauses are satisfiable.*

Note that there are several, related formulations that all go under the name of PCP theorem; the formulation above concerns the so-called "hardness of approximation". A formulation via Hamiltonians which is more suitable for our purposes is the following:

Theorem 2.2 (PCP theorem - Hamiltonian form)
There is a constant $\varepsilon > 0$ and a family of local Hamiltonians $\{H^{(n)}\}_{n\in\mathcal{I}}$ with each $H^{(n)}$ being a sum of terms which are diagonal in the computational basis and have norm bounded by a constant such that it is NP*-hard to decide which of the following is true (if promised that one of these holds):*

- *the ground state energy of $H^{(n)}$ is 0*
- *the ground state energy of $H^{(n)}$ is greater than εn.*

The connection between the two formulations lies in the fact that we can encode any 3SAT formula as a sum of diagonal terms which can be interpreted as a Hamiltonian; the ground state of this Hamiltonian being 0 then corresponds to the satisfiability of the 3SAT formula.

A first proof of the PCP theorem was given by Arora, Lund, Motwani, Sudan and Szegedy in 1998 [ALM$^+$98]; it was also proven in a different way in 2005 by Dinur [Din07] using expander graphs.

2.2 The qPCP conjecture and its relation to the NLTS conjecture

We can now ask whether a statement analogous the PCP theorem also holds in the quantum case. For this purpose, we simply omit the diagonality assumption on the local terms in the Hamiltonian form of the PCP theorem and take the "quantum analog" of NP, the complexity class QMA. This leads to the following conjecture:

Conjecture 2 (qPCP conjecture)
There is a constant $\varepsilon > 0$ and a family of local Hamiltonians $\{H^{(n)}\}_{n\in\mathcal{I}}$ with terms having norm bounded by a constant such that it is QMA*-hard to decide which of the following is true (if promised that one of these holds):*

- *the ground state energy of $H^{(n)}$ is* 0
- *the ground state energy of $H^{(n)}$ is greater than εn.*

We now establish the connection between the NLTS and the qPCP conjecture:

Theorem 2.3 (qPCP $\Rightarrow$ NLTS)
Assuming that QMA $\neq$ NP*, the validity of the qPCP conjecture implies the validity of the NLTS conjecture.*

Proof. Suppose that the NLTS conjecture does not hold. This means that for every $\varepsilon > 0$, all families of local Hamiltonians $\{H^{(n)}\}_{n\in\mathcal{I}}$ with ground state energy 0 (where $\mathcal{I} \subset \mathbb{N}$ is infinite) and all $f \colon \mathbb{N} \to \mathbb{N}$, the following holds: if we are given some circuit of depth d, there exists some $n \geq f(d)$ such that

$$\langle 0^n | U^\dagger H^{(n)} U | 0^n \rangle \leq \varepsilon n. \tag{1}$$

The circuit U can now serve as a *classical* witness from which one can efficiently compute the energy classically due to the constant depth of U and verify that the Hamiltonian indeed satisfies (1). This would place the decision problem inside NP $\subseteq$ QMA, a contradiction to the qPCP conjecture as long as QMA $\neq$ NP. □

3 Discussion

While the general NLTS conjecture stays unresolved, there has been some progress by proving slight variations of the conjecture. We briefly discuss two of these approaches in the following.

3.1 NLETS - "No Low Error Trivial States"

The first of these approaches is due to Eldar and Harrow [EH17]; they prove a weaker NLTS version that considers errors instead of violations.

Definition 3.1 (Ground-state impostors)
Let H be a Hamiltonian on n qubits. A quantum state ρ is an ε-impostor for H if there exists a set $S \subseteq [n]$, $|S| \geq (1-\varepsilon)n$ and a ground state σ such that $\rho_S = \sigma_S$.

Definition 3.2 (d-trivial states)
An n-qubit quantum state $|\psi\rangle$ is called d-trivial if there exists a local unitary quantum circuit U with depth d such that

$$|\psi\rangle = U|0^n\rangle.$$

Definition 3.3 (NLETS [EH17])
*A family of local Hamiltonians $\{H_n\}_{n\in\mathbb{N}}$ has the **NLETS property** if there exists $\varepsilon > 0$ and a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that the following holds: for any d and $n \geq f(d)$, if $\{\rho_n\}$ is any family of ε-impostor states for $\{H_n\}_{n\in\mathbb{N}}$, then $\{\rho_n\}$ is not d-trivial.*

Theorem 3.4 (NLETS [EH17])
There exists a family of local Hamiltonians with the NLETS property.

3.2 NL$\mathbb{Z}_2$TS

A recent approach by Bravyi, AK, König and Tang [BKKT19] is to look at a symmetry-protected version of the NLTS conjecture. The following definition gives the precise meaning of this term.

Definition 3.5 ($\mathbb{Z}_2$-symmetry)
In the following, let X denote the Pauli-operator.

- *A Hamiltonian $H^{(n)}$ on n qubits is said to be $\mathbb{Z}_2$-symmetric if it commutes with $X^{\otimes n}$, i.e.,*

$$[H^{(n)}, X^{\otimes n}] = 0.$$

- *A quantum circuit U on n qubits is said to be $\mathbb{Z}_2$-symmetric if it commutes with $X^{\otimes n}$, i.e.,*

$$[U, X^{\otimes n}] = 0$$

- *A state ψ on n qubits is said to be $\mathbb{Z}_2$-symmetric if it is an eigenstate of $X^{\otimes n}$ associated with eigenvalue ± 1, i.e.,*

$$X^{\otimes n}\psi = \pm\psi.$$

Using this approach, we were able to show the following version where every occurrence of Hamiltonian, circuit and state was replaced by its $\mathbb{Z}_2$-symmetric version:

Theorem 3.6 (NL$\mathbb{Z}_2$TS)
There exist constants $\varepsilon, c > 0$ and a family of $\underline{\mathbb{Z}_2\text{-symmetric}}$ local Hamiltonians $\{H_n\}_{n\in\mathcal{I}}$ such that H_n has ground state energy 0 *for any $n \in \mathcal{I}$ while*

$$\langle\phi|U^\dagger H_n U|\phi\rangle > \varepsilon n$$

for any $\underline{\mathbb{Z}_2\text{-symmetric}}$ depth-d circuit U composed of two-qubit gates, any $\underline{\mathbb{Z}_2\text{-symmetric}}$ product state ϕ, and any $n \geq 2^{cd}$, $n \in \mathcal{I}$.

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