Bell inequalities without input

In the standard Bell setting we have two observers (Alice, Bob) who share a physical system. Dependent on the input (i,j) they will provide an output (a,b). The set of conditional probability distributions of such systems which can be described by a local hidden variable model is given by the convex set

$$\mathcal{P}_{LHV} := \{P_{AB|IJ}(a,b|i,j) \mid \exists \text{ prob. measure } \mu(\lambda), \text{ cond. distributions } P_{A|I\Lambda}, P_{B|J\Lambda} : \\ P_{AB|IJ}(a,b|i,j) = \int d\mu(\lambda)\; P_{A|I\Lambda}(a|i\lambda) P_{B|J\Lambda}(b|j\lambda)\}, \tag{1}$$

whereas the set of probability distributions describing a general quantum system in the above setting is given by

$$\mathcal{P}_{Q_{input}} := \{P_{AB|IJ}(a,b|i,j) \mid \exists \text{ state } \rho_{AB}, \text{ POVMs } \{M_a^i\}_a, \{M_b^j\}_b : \\ P_{AB|IJ}(a,b|i,j) = \operatorname{tr}((M_a^i \otimes M_b^j)\rho_{AB})\}. \tag{2}$$

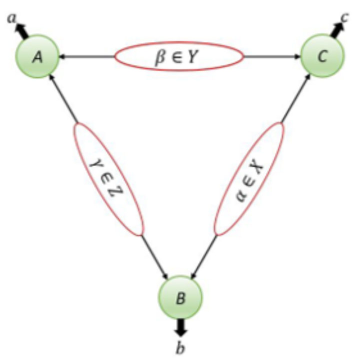
We have $\mathcal{P}_{LHV} \subset \mathcal{P}_{Q_{input}}$. Furthermore, there exist triples $(l, c, \widetilde{P}_{AB|IJ})$, where l is a linear function, c is a constant and $\widetilde{P}_{AB|IJ} \in \mathcal{P}_{Q_{input}}$, such that

$$l(P_{AB|IJ}) \leq c \quad \forall P_{AB|IJ} \in \mathcal{P}_{LHV} \tag{3}$$

$$\text{and } \exists \widetilde{P}_{AB|IJ} \in \mathcal{P}_{Q_{input}} : l(\widetilde{P}_{AB|IJ}) > c.$$

Thus, $\mathcal{P}_{LHV} \neq \mathcal{P}_{Q_{input}}$. Inequalities in the style of (3) are called Bell inequalities. Quantum non-locality is "traditionally" expressed by Bell inequality violations. As a conclusion we get $\mathcal{P}_{LHV} \subsetneq \mathcal{P}_{Q_{input}}$.

Now we consider a triangle network with three observers (Alice, Bob, Charlie). Each pair of observers is connected by a source, providing a shared physical system. Importantly, the three sources are assumed to be independent of each other. Each observer produces an output (a, b, c) based on the received physical resources. In contrast to the above standard Bell setting, the observers do not receive any input.



In the triangle network without input the set of classical probability distributions $\mathcal{P}_{classical}$ and the set of quantum probability distributions $\mathcal{P}_Q$ are given by

$$\begin{aligned}\mathcal{P}_{classical} \coloneqq \{&P_{ABC}(a,b,c) \mid \exists \text{ prob. measures } \mu_1,\mu_2,\mu_3, \text{ cond. distr.} P_{A|\beta\Gamma}, P_{B|\alpha\Gamma}, P_{C|\alpha\beta} : \\ &P_{ABC}(a,b,c) = \int d\mu_1(\alpha)d\mu_2(\beta)d\mu_3(\gamma)\; P_{A|\beta\Gamma}(a|\beta,\gamma)P_{B|\alpha\Gamma}(b|\alpha,\gamma)P_{C|\alpha\beta}(c|\alpha,\beta)\}\end{aligned} \tag{4}$$

and

$$\begin{aligned}\mathcal{P}_Q \coloneqq \{&P_{ABC}(a,b,c) \mid \exists \text{ states } \rho_{A_1B_1}, \rho_{B_2C_1}, \rho_{A_2C_2},\ \text{POVMs } \{M_a^{A_1A_2}\}_a, \{M_b^{B_1B_2}\}_b, \{M_c^{C_1C_2}\}_c : \\ &P_{ABC}(a,b,c) = \mathrm{tr}((M_a^{A_1A_2} \otimes M_b^{B_1B_2} \otimes M_c^{C_1C_2})(\rho_{A_1B_1} \otimes \rho_{B_2C_1} \otimes \rho_{A_2C_2}))\}.\end{aligned} \tag{5}$$

Note that $\mathcal{P}_{classical}$ is not a convex set, contrary to $\mathcal{P}_{LHV}$. This makes the characterization of $\mathcal{P}_{classical}$ much more difficult. In (Gisin et al., 2019) it is shown how to derive strong constraints on classical correlations in the triangle network by inflating it to a hexagon network. For the set of distributions $\mathcal{P}_{NSI}$ fulfilling these constraints we have $\mathcal{P}_{classical} \subset \mathcal{P}_{NSI}$.

Theorem 1. *For a given probability distribution P_{ABC} of the triangle network without input it is possible to determine whether it is compatible with the NSI constraints given in (Gisin et al., 2019). If $P_{ABC} \notin \mathcal{P}_{NSI}$, it follows that $P_{ABC} \notin \mathcal{P}_{classical}$.*

Furthermore, it is shown that some NSI bounds are tight by constructing explicit classical models that saturate the constraints. Whether $\mathcal{P}_{classical} = \mathcal{P}_{NSI}$ or $\mathcal{P}_{classical} \subsetneq \mathcal{P}_{NSI}$ remains an open problem.

Focussing on the relation between $\mathcal{P}_{classical}$ and $\mathcal{P}_Q$, we have $\mathcal{P}_{classical} \subset \mathcal{P}_Q$. We show $\mathcal{P}_{classical} \subsetneq \mathcal{P}_Q$ by constructing the following family of quantum distributions $P_Q(a,b,c)$ with properties a classical distribution can not have, cf. (Renou et al., 2019).
Consider a triangle quantum network with observers Alice, Bob and Charlie as sketched above. Let each source produce the same pure maximally entangled state

$$|\psi_\gamma\rangle_{A_\gamma B_\gamma} = |\psi_\alpha\rangle_{B_\alpha C_\alpha} = |\psi_\beta\rangle_{A_\beta C_\beta} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{6}$$

where each party receives two independent qubit subsystems, e.g. Alice receives A_β and A_γ. Furthermore, let each party perform a projective quantum measurement in the same basis given by

$$\begin{aligned} |\uparrow\rangle &= |01\rangle\,, \qquad & |\chi_0\rangle &= u\,|00\rangle + v\,|11\rangle\,, \\ |\downarrow\rangle &= |10\rangle\,, \qquad & |\chi_1\rangle &= u\,|00\rangle - v\,|11\rangle\,, \end{aligned} \tag{7}$$

with $u^2 + v^2 = 1$ and $0 < v < u < 1$. For Alice we get $\{|\phi_a\rangle_{A_\beta A_\gamma}\}$, where $\phi_a \in \{\uparrow, \downarrow, \chi_0, \chi_1\}$. The statistics of the experiment are given by

$$P_Q(a,b,c) = |\,\langle\phi_a|\,\langle\phi_b|\,\langle\phi_c|\,|\psi_\gamma\rangle\,|\psi_\alpha\rangle\,|\psi_\beta\rangle\,|^2. \tag{8}$$

Theorem 2. *$P_Q(a, b, c)$ can not be reproduced by any classical model of the triangle network when $u^2_{max} < u^2 < 1$, where $u^2_{max} = \frac{-3+(9+6\sqrt{2})^{2/3}}{2(9+6\sqrt{3})^{1/3}} \approx 0.785$.*

Hence, we have $P_Q(a, b, c) \subset \mathcal{P}_Q$, but $P_Q(a, b, c) \nsubseteq \mathcal{P}_{classical}$ for u chosen as above.

References

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Renou, M.-O., Bäumer, E., Boreiri, S., Brunner, N., Gisin, N., & Beigi, S. (2019). Genuine quantum nonlocality in the triangle network. *Physical review letters*, *123*(14), 140401.