# Dimensionality reduction of SDPs through sketching

Daniel Stilck França

Technische Universität München

MUNICH QUANTUM CENTER

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Joint work with Andreas Bluhm

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# Semidefinite Programs (SDPs)

Semidefinite programs are constrained optimization problems of the form:

$$\begin{array}{lll} \text{maximize} & \mathrm{tr}\,(AX) & \\ \text{subject to} & \mathrm{tr}\,(B_i X) \leq \gamma_i, & i \in [m] \\ & X \geq 0, & \end{array}$$

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They can be written in many equivalent forms. Will call this one *sketchable* SDP. Any SDP can be formulated in this form.

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- Bad news: scaling still prohibitive for high dimensional problems! Especially when it comes to memory. Try running an SDP with $D \sim 10^3$ on your laptop and you will already run out of memory.
- Need techniques to solve larger problems. Ideally using available solvers.

# Sketch of Idea

- Apply a positive linear map $\Phi : \mathcal{M}_D \to \mathcal{M}_d$ to constraints s.t. $\mathrm{tr}\left(\Phi(B_i)\Phi(X^*)\right) \approx \mathrm{tr}\left(B_i X^*\right)$ holds with high probability. Here $X^*$ is a solution of the SDP.

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- Solve the SDP defined by $\Phi(B_i)$ and show that its value is not far from the value of the original problem.
- If $d << D$ and computing $\Phi(B_i)$ is cheap, then this gives a computational advantage.

# SDPs cannot be sketched

**Theorem (SDPs cannot be sketched)**

*Let $\Phi : \mathcal{M}_{2D} \to \mathbb{R}^d$ be a random linear map such that for all sketchable SDPs there exists an algorithm which allows us to estimate the value of an SDP up to a constant factor $1 \leq \tau < \frac{2}{\sqrt{3}}$ given the sketch $\{\Phi(A), \Phi(B_1), \ldots, \Phi(B_m)\}$ with probability at least $9/10$. Then $d = \Omega(D^2)$.*

# SDPs cannot be sketched

**Theorem (Not all SDPs can be sketched)**

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# Johnson-Lindenstrauss transforms

**Definition (Johnson-Lindenstrauss transform)**

*A random matrix $S \in \mathcal{M}_{d,D}$ is a Johnson-Lindenstrauss transform (JLT) with parameters $(\epsilon, \delta, k)$ if with probability at least $1 - \delta$, for any $k$-element subset $V \subseteq \mathbb{K}^D$, for all $v, w \in V$ it holds that*

$$|\langle Sv, Sw\rangle - \langle v, w\rangle| \leq \epsilon \|v\|_2 \|w\|_2.$$

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- Example: $S = \frac{1}{\sqrt{d}} R \in \mathcal{M}_{d,D}$, where the entries of $R$ are i.i.d. standard Gaussian random variables. If $d = \Omega(\epsilon^{-2}\log(k\delta^{-1}))$, then $S$ is an $(\epsilon, \delta, k)$-JLT.

# Sketching the HS scalar product

**Lemma (Sketching Hilbert-Schmidt Scalar Product)**

*Let $B_1, \ldots, B_m \in \mathcal{M}_D$ and $S \in \mathcal{M}_{d,D}$ be an $(\epsilon, \delta, k)$-JLT with $\epsilon \leq 1$ and $k$ such that*

$$k \geq \sum_{i=1}^{m} rank\,(B_i)\,.$$

*Then, with probability at least $1 - \delta$,*

$$\forall i, j \in [m] : |tr\left(SB_iS^TSB_jS^T\right) - tr(B_iB_j)| \leq 3\epsilon \|B_i\|_1 \|B_j\|_1$$

# Bad scaling

$$\left|\operatorname{tr}\left(SB_iS^TSB_jS^T\right) - \operatorname{tr}\left(B_iB_j\right)\right| \leq 3\epsilon\|B_i\|_1\|B_j\|_1$$

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- Scaling with $\|\cdot\|_1$ is undesirable. Normal JLT gives scaling with $\|\cdot\|_2$.
- Proof of the inequality admittedly crude. Can we improve the inequality?

# No Johnson-Lindenstrauss with positive maps

**Theorem (No Johnson-Lindenstrauss with positive maps)**

*Let $\Phi : \mathcal{M}_D \to \mathcal{M}_d$ be a random positive map such that with strictly positive probability for any $Y_1, \ldots Y_{D+1} \in \mathcal{M}_D$ and $0 < \epsilon < \frac{1}{4}$ we have*

$$|tr\left(\Phi(Y_i)^T\Phi(Y_j)\right) - tr\left(Y_i^T Y_j\right)| \leq \epsilon \|Y_i\|_2 \|Y_j\|_2.$$

*Then $d = \Omega(D)$.*

# The Algorithm

- Assumptions: uniform bounds on $\|A\|_1, \|B_1\|_1, \ldots, \|B_m\|_1$ and $\|X^*\|_1$, where $X^*$ is an optimal point of the SDP. Standard regularity assumptions on SDP.

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- Assumptions: uniform bounds on $\|A\|_1, \|B_1\|_1, \ldots, \|B_m\|_1$ and $\|X^*\|_1$, where $X^*$ is an optimal point of the SDP. Standard regularity assumptions on SDP.

Consider the sketchable SDP of dimension $D$:

$$\begin{aligned} \text{maximize} \quad & \operatorname{tr}(AX) \\ \text{subject to} \quad & \operatorname{tr}(B_i X) \leq \gamma_i, \qquad i \in [m] \\ & X \geq 0. \end{aligned}$$

# The Algorithm

Now pick a $(\epsilon, \delta, k)$ JL-transform $S \in \mathcal{M}_{d,D}$, where

$$k \geq \text{rank}(X^*) + \text{rank}(A) + \sum_{i=1}^{m} \text{rank}(B_i),$$

and consider the SDP of dimension $d$:

$$\begin{aligned} &\text{maximize} && \text{tr}\left(SAS^T Y\right) \\ &\text{subject to} && \text{tr}\left(SB_iS^T Y\right) \leq \gamma_i, \qquad i \in [m] \\ &&& Y \geq 0. \end{aligned}$$

# The Algorithm

Relax the constraints:

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\begin{aligned}
\text{maximize} \quad & \operatorname{tr}\left(SAS^T Y\right) \\
\text{subject to} \quad & \operatorname{tr}\left(SB_iS^T Y\right) \leq \gamma_i + 3\epsilon \|B_i\|_1 \|X^*\|_1, \qquad i \in [m] \\
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Call this SDP the sketched SDP! Solve this SDP!

# The Algorithm

Relax the constraints:

$$\begin{aligned} \text{maximize} \quad & \mathrm{tr}\left(SAS^TY\right) \\ \text{subject to} \quad & \mathrm{tr}\left(SB_iS^TY\right) \leq \gamma_i + 3\epsilon\|B_i\|_1\|X^*\|_1, \qquad i \in [m] \\ & Y \geq 0. \end{aligned}$$

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- Bound on HS scalar product gives that $SX^*S^T$ is a feasible point of the relaxed problem with probability $\geq 1 - \delta$.
- We have $\operatorname{tr}\left(SAS^T SX^*S^T\right) \geq \operatorname{tr}\left(AX^*\right) - 3\epsilon\|X^*\|_1\|A\|_1$.

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- We have $\operatorname{tr}\left(SAS^TSX^*S^T\right) \geq \operatorname{tr}\left(AX^*\right) - 3\epsilon\|X^*\|_1\|A\|_1$.
- But $\operatorname{tr}\left(AX^*\right) = \alpha$ is the value of the sketchable SDP! We therefore obtain $\alpha_S + 3\epsilon\|X^*\|_1\|A\|_1 \geq \alpha$, where $\alpha_S$ is the value of the sketched SDP.

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# Upper bound through sketch

**Theorem**

*Let $A, B_1, \ldots, B_m \in \mathcal{M}_D^{sym}$, $\eta, \gamma_1, \ldots, \gamma_m \in \mathbb{R}$ and $\epsilon > 0$. Denote by $\alpha$ the value of the sketchable SDP and assume it is attained at an optimal point $X^*$ which satisfies $tr(X^*) \leq \eta$. Moreover, let $S \in \mathcal{M}_{d,D}$ be an $(\epsilon, \delta, k)$-JLT, with*

$$k \geq rank(X^*) + rank(A) + \sum_{i=1}^{m} rank(B_i).$$

*Let $\alpha_S$ be the value of the sketched SDP defined by $A$, $B_i$ and $S$. Then*

$$\alpha_S + 3\epsilon\eta\|A\|_1 \geq \alpha$$

*with probability at least $1 - \delta$.*

# Lower Bound

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- Depends on how stable your SDP is!

# Lower Bound

Let $Y^*$ be an optimal point of your sketched SDP. That is, a solution of:

$$\begin{aligned} \text{maximize} \quad & \operatorname{tr}\left(SAS^T Y\right) \\ \text{subject to} \quad & \operatorname{tr}\left(SB_iS^T Y\right) \leq \gamma_i + 3\epsilon \|B_i\|_1 \|X^*\|_1, \qquad i \in [m] \\ & Y \geq 0 \end{aligned}$$

# Lower Bound

By the cyclicity of the trace, $S^T Y^* S$ is a feasible point of

$$\begin{aligned} \text{maximize} \quad & \operatorname{tr}(AX) \\ \text{subject to} \quad & \operatorname{tr}(B_i X) \leq \gamma_i + 3\epsilon \|B_i\|_1 \|X^*\|_1, \qquad i \in [m] \\ & X \geq 0 \end{aligned}$$

with value $\alpha_S$.

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with value $\alpha_S$. This is just a perturbed verision of the original SDP!

# Lower Bound for positive $\gamma_i$

**Theorem (Lower Bound in terms of $\alpha_S$)**

*For a sketchable SDP with $\gamma_i = 1$ and $\kappa = \max_{i\in[m]} \|B_i\|_1$, we have that*

$$\frac{\alpha_S}{1+\nu} \leq \alpha,$$

*where $\nu = 3\epsilon\eta\kappa$. Moreover, denoting by $X_S^*$ an optimal point of the sketched SDP, we have that $\frac{1}{1+\nu} S^T X_S^* S$ is a feasible point of the sketchable SDP that attains this lower bound.*

# Summary

**Theorem**

*For a sketchable SDP with $\gamma_i = 1$ and $\kappa = \max_{i \in [m]} \|B_i\|_1$, we have that*

$$\frac{\alpha_S}{1+\nu} \leq \alpha \leq \alpha_S + 3\epsilon\eta\|A\|_1,$$

*where $\nu = 3\epsilon\eta\kappa$.*

# Complexity and Memory Considerations

- Assuming $\|A\|_1, \|B_1\|, \ldots, \|B_m\|_1, \|X^*\|_1 = \mathcal{O}(1)$ and $\epsilon, \delta, \zeta$ fixed we obtain:

**Theorem**

*Let $A, B_1, \ldots, B_m \in \mathcal{M}_D^{sym}$, of a sketchable SDP be given. Furthermore, let $\mathrm{SDP}(m, d)$ be the complexity of solving a sketchable SDP of dimension $d$ and $m$ constraints up to some given precision. Then a number of*

$$\mathcal{O}(D^2 m \log(k)) + \mathrm{SDP}(m, \log(k)))$$

*operations is needed to generate and solve the sketched SDP, where $k \leq (m+2)D^2$ is defined as before.*

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- Assuming $\epsilon, \delta$ fixed, sketching gives a speedup as long as the complexity of solving the SDP directly is $\Omega(mD^{2+\mu})$, where $\mu > 0$.

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- Need only to store $\mathcal{O}(m\epsilon^{-4}\log(mk/\delta)^2)$ entries to solve the sketched problem.

# Uncertainty Relations

Given observables $A, B \in \mathcal{M}_D^{\text{sym}}$, consider uncertainty relations of the form

$$\operatorname{tr}\left(A^2\rho\right) + \operatorname{tr}\left(B^2\rho\right) \geq c$$

for all states $\rho$ s.t. $\operatorname{tr}(A\rho) \in (a-\epsilon, a+\epsilon)$ and $\operatorname{tr}(B\rho) \in (b-\epsilon, b+\epsilon)$.

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for all states $\rho$ s.t. $\operatorname{tr}(A\rho) \in (a-\epsilon, a+\epsilon)$ and $\operatorname{tr}(B\rho) \in (b-\epsilon, b+\epsilon)$. Finding the optimal $c$ can easily be cast as an SDP.

# Uncertainty Relations

$$\begin{aligned}
\text{minimize} \quad & \operatorname{tr}\left((A^2 + B^2)X\right) \\
\text{subject to} \quad & \operatorname{tr}(AX) \leq a \pm \epsilon, \\
& \operatorname{tr}(BX) \leq b \pm \epsilon, \\
& \operatorname{tr}(X) = 1, \\
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- If $\|B\|_1, \|A\|_1 = \mathcal{O}(1)$ and their nonzero spectrum "flat", we can show $\|X^*\|_1 = \mathcal{O}(1)$.
- Example: $A, B$ of fixed rank and nonzero spectrum contained in some compact interval.

| $D$ | $d$ | Value | Error L.B. | M.R.T. Sketchable [s] | M.R.T Sketch [s] |
|---|---|---|---|---|---|
| 200 | 50 | 0.0928 | 0.0429 | 6.73 | 0.663 |
| 200 | 100 | 0.0897 | 0.0401 | 6.51 | 1.336 |
| 500 | 100 | 0.0353 | 0.0181 | 96.5 | 1.35 |
| 500 | 200 | 0.0364 | 0.0152 | 96.4 | 6.81 |

Table: For each combination of the sketchable dimension ($D$) and dimension of the sketch ($d$) we have generated 40 instances of the uncertainty relation SDP. Here "M.R.T." stands for mean running time, "L.B." stands for lower bound we obtained from the sketch. The column "Value" stands for the optimal value of the sketchable SDP.

Thanks!