Logarithmic Sobolev Inequalities for Entropy Production

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Joint work with Alexander Müller-Hermes and Michael M. Wolf

arXiv:1505.04678

Outline

- Definition and applications of the Logarithmic Sobolev 1 constant.

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- The Logarithmic Sobolev 1 constant for depolarizing semigroups and applications to the concavity of the von Neumann entropy.

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- The Logarithmic Sobolev 1 constant for depolarizing semigroups and applications to the concavity of the von Neumann entropy.
- The Logarithmic Sobolev 2 constant, hypercontractivity and LS inequalities that tensorize with applications to the entropy production.

Logarithmic Sobolev 1 Constant

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$$D\left(e^{t\mathcal{L}}\rho||\sigma\right) \leq e^{-2\alpha_1 t} D\left(\rho||\sigma\right)$$

with $D\left(\rho||\sigma\right) = \mathrm{tr}[\rho(\log(\rho) - \log(\sigma))]$.

Given a primitive Liouvillian $\mathcal{L} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ with stationary state $\sigma \in \mathcal{D}_d^+$ we want to estimate the convergence in the relative entropy:

$$D\left(e^{t\mathcal{L}}\rho||\sigma\right) \leq e^{-2\alpha_1 t} D\left(\rho||\sigma\right) \qquad (1)$$

with $D\left(\rho||\sigma\right) = \text{tr}[\rho(\log(\rho) - \log(\sigma))]$.
The largest α_1 s.t. (1) holds for all $t > 0$ is the Logarithmic Sobolev 1 constant.

Entropy Production

For $S(\rho) = -\text{tr}[\rho \log(\rho)]$ the von Neumann entropy and doubly stochastic Liouvillians ($\mathcal{L}(\mathbb{1}) = \mathcal{L}^*(\mathbb{1}) = 0$), a LS-1 inequality is equivalent to:

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Provides a way of quantifying the production of entropy by the semigroup.

Why you should care

LS inequalities have already found many applications, such as:

1. If we have a family of Liouvillians defined on a lattice that have a LS constant which does not scale with size of the system, this implies:
   - Strong notion of stability of observables w.r.t. perturbations of the Liouvillian[1].

[1] T. S. Cubitt et al. "Stability of local quantum dissipative systems". In: *ArXiv e-prints* (Mar. 2013). arXiv:1303.4744 [quant-ph]

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    - Area law and exponential decay of correlations for the stationary state. [2][3].

[2] M. J. Kastoryano and J. Eisert. "Rapid mixing implies exponential decay of correlations". In: *Journal of Mathematical Physics* 54.10 (Oct. 2013), p. 102201. DOI: 10.1063/1.4822481. arXiv:1303.6304 [quant-ph]

[3] F. G. S. L. Brandao et al. "Area law for fixed points of rapidly mixing dissipative quantum systems". In: *ArXiv e-prints* (May 2015). arXiv:1505.02776 [quant-ph]

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2. These are all consequence of rapid mixing:

$$||e^{t\mathcal{L}}(\rho) - \sigma||_1 \leq e^{-\alpha_1 t}\sqrt{2\log\left(\sigma_{\min}^{-1}\right)}$$

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3. Refinements of entropic inequalities.
4. Analysis of the lifetime of quantum memories.

Differential Formulation

We can express the LS-1 constant as:

$$\alpha_1(\mathcal{L}) = \inf_{\rho \in \mathcal{D}_d^+} \frac{\mathrm{tr}[\mathcal{L}(\rho)(\log(\sigma) - \log(\rho))]}{D(\rho \| \sigma)}$$

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Hard to compute analytically! Only known for doubly stochastic, reversible qubit Liouvillians!

LS-1 Constant for the Depolarizing Channel

Using techniques from fractional programming, we have computed this constant for the depolarizing channels $\mathcal{L}_\sigma(\rho) = \mathrm{tr}(\rho)\sigma - \rho$, $\sigma \in \mathcal{D}_d^+$ arbitrary.

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$$\alpha_1(\mathcal{L}_\sigma) = \min_{x \in [0,1]} \frac{1}{2}\left(1 + \frac{D_2(\sigma_{\min}||x)}{D_2(x||\sigma_{\min})}\right)$$

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where D_2 is the binary relative entropy. We also have:

$$1 \geq \alpha_1(\mathcal{L}_\sigma) \geq \frac{1}{2}\left(1 + \sqrt{\sigma_{\min}(1-\sigma_{\min})}\right)$$

It follows from the last result that for $\rho, \sigma \in \mathcal{D}_d$ and $q \in [0,1]$ we have

$$S((1-q)\sigma + q\rho) - (1-q)S(\sigma) - qS(\rho) \geq \max \begin{cases} q(1-q^{c(\sigma)})D(\rho\|\sigma) \\ (1-q)(1-(1-q)^{c(\rho)})D(\sigma\|\rho) \end{cases},$$

with

$$c(\sigma) = \min_{x\in[0,1]} \frac{D_2(\sigma_{\min}\|x)}{D_2(x\|\sigma_{\min})}$$

and $c(\rho)$ defined in the same way.

Similar Result by Kim and Ruskai

We have[4]:

$$S((1-q)\sigma + q\rho) - (1-q)S(\sigma) - qS(\rho) \geq \frac{(1-q)q}{2}||\rho - \sigma||_1^2$$

[4] Isaac Kim and Mary Beth Ruskai. "Bounds on the concavity of quantum entropy". In: *Journal of Mathematical Physics* 55.9, 092201 (2014), pp. –

Comparison with Similar Results

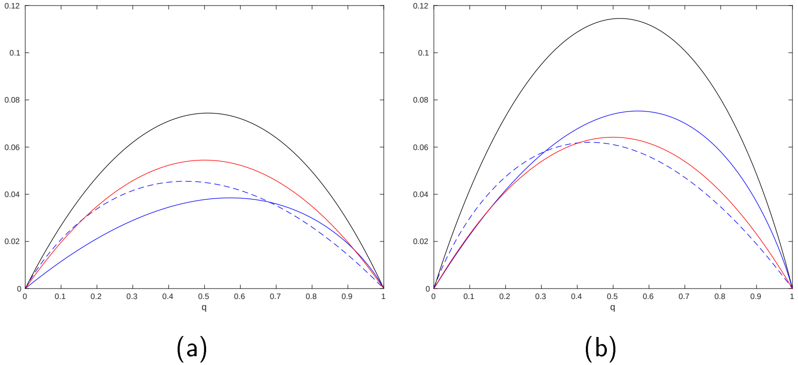

Figure : Comparison of bound the bound by Kim(red), ours (blue) and the exact value $S((1-q)\sigma + q\rho) - (1-q)S(\sigma) - qS(\rho)$ (black).

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Entropy Production and Hypercontractivity

- It is desirable to have lower bounds on $\alpha_1(\mathcal{L})$ that are easier to evaluate.
- We will focus on doubly stochastic, reversible Liouvillians ($\mathcal{L} = \mathcal{L}^*, \mathcal{L}(\mathbb{1}) = 0$).
- For quantum memories it is desirable to have bounds that tensorize, that is $\alpha_1(\mathcal{L}^{(n)}) \geq c$ and $\mathcal{L}^{(n)}$ the generator of $(e^{t\mathcal{L}})^{\otimes n}$.

The LS-2 constant of $\mathcal{L}$ is defined as the optimal $\alpha_2 > 0$ s.t. for all $X \in \mathcal{M}_d^+$ and $t > 0$

$$d^{\frac{1}{2}-\frac{1}{p(t)}} \frac{||e^{t\mathcal{L}}X||_{p(t)}}{||X||_2} \leq 1$$

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Interpretation: larger p emphasizes the "peaks" in the spectrum of X. If we have a small p-norm with p large, this means the spectrum is "flat".

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Interpretation: larger p emphasizes the "peaks" in the spectrum of X. If we have a small p-norm with p large, this means the spectrum is "flat".
$\alpha_1(\mathcal{L}) \geq \alpha_2(\mathcal{L})$
Easier to handle!

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$$||\mathcal{L}||\alpha_2\left(\mathcal{L}_{\frac{\mathbb{1}}{d}}\right) \geq \alpha_2\left(\mathcal{L}\right) \geq \lambda\alpha_2\left(\mathcal{L}_{\frac{\mathbb{1}}{d}}\right)$$

where λ is the spectral gap of $\mathcal{L}$ (second smallest eigenvalue of $-\mathcal{L}$) .

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where λ is the spectral gap of $\mathcal{L}$ (second smallest eigenvalue of $-\mathcal{L}$) .

- This inequality tensorizes.
- A bound for the depolarizing channel gives a universal lower bound in terms of the spectral gap!

- Use group theoretic techniques to relate the LS-2 constant of the depolarizing channel to the LS-2 of a classical Markov chain with known LS-2 constant.

[5] Kristan Temme, Fernando Pastawski, and Michael J Kastoryano. "Hypercontractivity of quasi-free quantum semigroups". In: *Journal of Physics A: Mathematical and Theoretical* 47.40 (2014)

Lower Bound for Depolarizing Channels

- Use group theoretic techniques to relate the LS-2 constant of the depolarizing channel to the LS-2 of a classical Markov chain with known LS-2 constant.
- These stay invariant under taking tensor powers, so we obtain:

$$\alpha_2\left(\mathcal{L}^{(n)}_{\frac{\mathbb{1}}{d}}\right) \geq \frac{\left(1-2d^{-2}\right)}{\log(3)\log(d^2-1)+2\left(1-2d^{-2}\right)}$$

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- Improves upon previous bounds[5] and has the right order of magnitude.

[5] Kristan Temme, Fernando Pastawski, and Michael J Kastoryano. "Hypercontractivity of quasi-free quantum semigroups". In: *Journal of Physics A: Mathematical and Theoretical* 47.40 (2014)

For any doubly stochastic Liouvillian it follows that:

$$\alpha_2\left(\mathcal{L}^{(n)}\right) \geq \lambda\frac{\left(1-2d^{-2}\right)}{\log(3)\log(d^2-1)+2\left(1-2d^{-2}\right)}$$

λ is its spectral gap.

General Doubly Stochastic Liouvillians

In terms of the entropy production, we have that:

$$S\left(\left(e^{t\mathcal{L}}\right)^{\otimes n}\rho\right) - S(\rho) \geq (1 - e^{-2\alpha t})(n\log(d) - S(\rho))$$

with $\alpha = \lambda\frac{\left(1-2d^{-2}\right)}{\log(3)\log(d^2-1)+2(1-2d^{-2})}$

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This inequality can be used to analyze quantum memories subjected to doubly stochastic noise.

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- The potential quality of this bound decreases as the local dimension increases, as made explicit by the depolarizing semigroups.

Summary

- LS inequalities are a powerful framework to show entropic inequalities and rapid mixing.
- It is difficult to obtain analytical results. Hypercontractivity is a valuable tool to obtain lower bounds, especially for product channels.
- The potential quality of this bound decreases as the local dimension increases, as made explicit by the depolarizing semigroups.
- The entropy always increases exponentially fast under local, primitive and doubly stochastic noise.

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Thanks!