Logarithmic Sobolev Inequalities for Entropy Production

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Joint work with Alexander Müller-Hermes and Michael M. Wolf

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Definition and applications of the Logarithmic Sobolev 1 constant.
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The Logarithmic Sobolev 1 constant for depolarizing semigroups and applications to the concavity of the von Neumann entropy.
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The Logarithmic Sobolev 1 constant for depolarizing semigroups and applications to the concavity of the von Neumann entropy.

The Logarithmic Sobolev 2 constant, hypercontractivity and LS inequalities that tensorize with applications to the entropy production.
Given a Liouvillian $L : \mathcal{M}_d \to \mathcal{M}_d$ with stationary state $\sigma \in \mathcal{D}_d^+$ we want to estimate the convergence in the relative entropy:
Given a Liouvillian $\mathcal{L} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ with stationary state $\sigma \in \mathcal{D}_d^+$ we want to estimate the convergence in the relative entropy:

$$D( e^{t\mathcal{L}} \rho \| \sigma ) \leq e^{-2\alpha_1 t} D( \rho \| \sigma )$$

with $D( \rho \| \sigma ) = \text{tr}[\rho(\log(\rho) - \log(\sigma))]$. 
Logarithmic Sobolev 1 Constant

Given a primitive Liouvillian $\mathcal{L} : \mathcal{M}_d \to \mathcal{M}_d$ with stationary state $\sigma \in \mathcal{D}_d^+$ we want to estimate the convergence in the relative entropy:

$$D \left( e^{t\mathcal{L}} \rho \| \sigma \right) \leq e^{-2\alpha_1 t} D \left( \rho \| \sigma \right) \quad (1)$$

with $D (\rho \| \sigma) = \text{tr}[\rho(\log(\rho) - \log(\sigma))]$.

The largest $\alpha_1$ s.t. (1) holds for all $t > 0$ is the Logarithmic Sobolev 1 constant.
For $S(\rho) = -\text{tr}[\rho \log(\rho)]$ the von Neumann entropy and doubly stochastic Liouvillians ($\mathcal{L}(\mathbb{1}) = \mathcal{L}^*(\mathbb{1}) = 0$), a LS-1 inequality is equivalent to:
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Provides a way of quantifying the production of entropy by the semigroup.
LS inequalities have already found many applications, such as:

1. If we have a family of Liouvillians defined on a lattice that have a LS constant which does not scale with size of the system, this implies:
   - Strong notion of stability of observables w.r.t. perturbations of the Liouvillian\(^1\).

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   - Area law and exponential decay of correlations for the stationary state.\(^2\)


Why you should care

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2. These are all consequence of rapid mixing:

\[ \| e^{t \mathcal{L}} (\rho) - \sigma \|_1 \leq e^{-\alpha_1 t} \sqrt{2 \log (\sigma_{\min}^{-1})} \]
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3. Refinements of entropic inequalities.

4. Analysis of the lifetime of quantum memories.
We can express the LS-1 constant as:

\[
\alpha_1 (\mathcal{L}) = \inf_{\rho \in \mathcal{D}_d^+} \frac{\text{tr}[\mathcal{L}(\rho)(\log(\sigma) - \log(\rho))]}{D(\rho || \sigma)}
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We can express the LS-1 constant as:

$$\alpha_1 (\mathcal{L}) = \inf_{\rho \in \mathcal{D}_d^+} \frac{\text{tr}[\mathcal{L}(\rho)(\log(\sigma) - \log(\rho))]}{D(\rho\|\sigma)}$$

Hard to compute analytically! Only known for doubly stochastic, reversible qubit Liouvillians!
Using techniques from fractional programming, we have computed this constant for the depolarizing channels $L_\sigma(\rho) = \text{tr}(\rho)\sigma - \rho$, $\sigma \in D^+_d$ arbitrary.
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\[
\alpha_1(\mathcal{L}_\sigma) = \min_{x \in [0,1]} \frac{1}{2} \left( 1 + \frac{D_2(\sigma_{\text{min}}\|x)}{D_2(x\|\sigma_{\text{min}})} \right)
\]

where \( D_2 \) is the binary relative entropy.
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\]

where \( D_2 \) is the binary relative entropy. We also have:

\[
1 \geq \alpha_1(\mathcal{L}_\sigma) \geq \frac{1}{2} \left( 1 + \sqrt{\sigma_{\text{min}}(1 - \sigma_{\text{min}})} \right)
\]

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Logarithmic Sobolev Inequalities for Entropy Production
It follows from the last result that for $\rho, \sigma \in \mathcal{D}_d$ and $q \in [0, 1]$ we have

$$S((1 - q)\sigma + q\rho) - (1 - q)S(\sigma) - qS(\rho) \geq \max\left\{ q(1 - q^{c(\sigma)})D(\rho\|\sigma) \\
(1 - q)(1 - (1 - q)^{c(\rho)})D(\sigma\|\rho) \right\},$$

with

$$c(\sigma) = \min_{x \in [0,1]} \frac{D_2(\sigma_{\text{min}}\|x)}{D_2(x\|\sigma_{\text{min}})}$$

and $c(\rho)$ defined in the same way.
Similar Result by Kim and Ruskai

We have\(^4\):

\[
S((1 - q)\sigma + q\rho) - (1 - q)S(\sigma) - qS(\rho) \geq \frac{(1 - q)q}{2} \|\rho - \sigma\|_1^2
\]

Comparison with Similar Results

Figure: Comparison of bound the bound by Kim (red), ours (blue) and the exact value $S((1-q)\sigma + q\rho) - (1-q)S(\sigma) - qS(\rho)$ (black).
It is desirable to have lower bounds on $\alpha_1(\mathcal{L})$ that are easier to evaluate.
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We will focus on doubly stochastic, reversible Liouvillians ($\mathcal{L} = \mathcal{L}^*, \mathcal{L}(\mathbb{1}) = 0$).

For quantum memories it is desirable to have bounds that tensorize, that is $\alpha_1(\mathcal{L}^{(n)}) \geq c$ and $\mathcal{L}^{(n)}$ the generator of $(e^{t\mathcal{L}})^{\otimes n}$.
The LS-2 constant of $\mathcal{L}$ is defined as the optimal $\alpha_2 > 0$ s.t. for all $X \in \mathcal{M}_d^+$ and $t > 0$

\[ d^{\frac{1}{2}} \left( \frac{1}{p(t)} \right) \frac{\| e^{t \mathcal{L}} X \|_{p(t)}}{\| X \|_2} \leq 1 \]

holds for $p(t) = 1 + e^{2\alpha_2 t}$. 

Interpretation: larger $p$ emphasizes the peaks in the spectrum of $X$. If we have a small $p$-norm with $p$ large, this means the spectrum is at.

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Interpretation: larger $p$ emphasizes the “peaks” in the spectrum of $X$. If we have a small $p$-norm with $p$ large, this means the spectrum is “flat”.
The LS-2 constant of $\mathcal{L}$ is defined as the optimal $\alpha_2 > 0$ s.t. for all $X \in \mathcal{M}_d^+$ and $t > 0$

$$d^{\frac{1}{2}} - \frac{1}{p(t)} \frac{\|e^{t\mathcal{L}}X\|_{p(t)}}{\|X\|_2} \leq 1$$

holds for $p(t) = 1 + e^{2\alpha_2 t}$. 

Interpretation: larger $p$ emphasizes the “peaks” in the spectrum of $X$. If we have a small $p$-norm with $p$ large, this means the spectrum is “flat”.

$\alpha_1 (\mathcal{L}) \geq \alpha_2 (\mathcal{L})$

Easier to handle!
Depolarizing channels again

Using a comparison technique, we show:

\[ \| L \| \alpha^2 (L^1d) \geq \lambda_{\alpha^2} (L^1d) \]

where \( \lambda \) is the spectral gap of \( -L \) (second smallest eigenvalue of \( -L \)).

This inequality tensorizes. An bound for the depolarizing channel gives a universal lower bound in terms of the spectral gap!
Using a comparison technique, we show:

\[ \|L\| \alpha_2 \left( L_{\frac{1}{d}} \right) \geq \alpha_2 (L) \geq \lambda \alpha_2 \left( L_{\frac{1}{d}} \right) \]

where \( \lambda \) is the spectral gap of \( L \) (second smallest eigenvalue of \( -L \)).
Using a comparison technique, we show:

\[ \|\mathcal{L}\| \alpha_2 \left( \mathcal{L}_{\frac{1}{d}} \right) \geq \alpha_2 (\mathcal{L}) \geq \lambda \alpha_2 \left( \mathcal{L}_{\frac{1}{d}} \right) \]

where \( \lambda \) is the spectral gap of \( \mathcal{L} \) (second smallest eigenvalue of \( -\mathcal{L} \)).

This inequality tensorizes.
Using a comparison technique, we show:

\[ \| \mathcal{L} \|_{\alpha_2} \left( \mathcal{L}_{1/d} \right) \geq \alpha_2 (\mathcal{L}) \geq \lambda \alpha_2 \left( \mathcal{L}_{1/d} \right) \]

where \( \lambda \) is the spectral gap of \( \mathcal{L} \) (second smallest eigenvalue of \( -\mathcal{L} \)).

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A bound for the depolarizing channel gives a universal lower bound in terms of the spectral gap!
Use group theoretic techniques to relate the LS-2 constant of the depolarizing channel to the LS-2 of a classical Markov chain with known LS-2 constant.

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These stay invariant under taking tensor powers, so we obtain:

$$\alpha^2 \left( \mathcal{L}_{\frac{1}{d}}^{(n)} \right) \geq \frac{(1 - 2d^{-2})}{\log(3) \log(d^2 - 1) + 2(1 - 2d^{-2})}$$

Use group theoretic techniques to relate the LS-2 constant of the depolarizing channel to the LS-2 of a classical Markov chain with known LS-2 constant.

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$$\alpha_2 \left( \mathcal{L}_{\frac{1}{d}}^{(n)} \right) \geq \frac{(1 - 2d^{-2})}{\log(3) \log(d^2 - 1) + 2(1 - 2d^{-2})}$$

Improves upon previous bounds\(^\text{5}\) and has the right order of magnitude.

For any doubly stochastic Liouvillian it follows that:

\[ \alpha_2 \left( L^{(n)} \right) \geq \lambda \frac{(1 - 2d^{-2})}{\log(3) \log(d^2 - 1) + 2 (1 - 2d^{-2})} \]

\( \lambda \) is its spectral gap.
In terms of the entropy production, we have that:

$$S((e^{tL})^\otimes n \rho) - S(\rho) \geq (1 - e^{-2\alpha t})(n \log(d) - S(\rho))$$

with $\alpha = \lambda \frac{(1-2d^{-2})}{\log(3) \log(d^2-1)+2(1-2d^{-2})}$.
In terms of the entropy production, we have that:

\[ S((e^{t\mathcal{L}})^n \rho) - S(\rho) \geq (1 - e^{-2\alpha t})(n \log(d) - S(\rho)) \]

with \( \alpha = \lambda \frac{(1-2d^{-2})}{\log(3) \log(d^2-1) + 2(1-2d^{-2})} \)

This inequality can be used to analyze quantum memories subjected to doubly stochastic noise.
LS inequalities are a powerful framework to show entropic inequalities and rapid mixing.
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LS inequalities are a powerful framework to show entropic inequalities and rapid mixing.

It is difficult to obtain analytical results. Hypercontractivity is a valuable tool to obtain lower bounds, especially for product channels.

The potential quality of this bound decreases as the local dimension increases, as made explicit by the depolarizing semigroups.

The entropy always increases exponentially fast under local, primitive and doubly stochastic noise.
Quantum logarithmic Sobolev inequalities and rapid mixing.

Hypercontractivity in noncommutative $L_p$-spaces.

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Thanks!